MATHEMATICS MAGAZINE

CONTENTS

VOLUME 41 • SEPTEMBER 1968 • NUMBER 4

MM

MATHEMATICS MAGAZINE

ROY DUBISCH, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the Editor-Elect, S. A. JENNINGS, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Buffalo, New York, bi-monthly except July–August. Ordinary subscriptions are: 1 year $4.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for $6.00. Single copies are 80¢.

Second class postage paid at Buffalo, New York and additional mailing offices.

A NOTE ON "INSTANT INSANITY"

T. A. BROWN, Rand Corporation

An interesting puzzle has recently appeared in toy stores under the name "Instant Insanity." It consists of four multicolored unit cubes. Each cube has its faces painted red, blue, white, and green; the exact coloring is indicated in Figure 1. The puzzle is to assemble these four cubes into a $1 \times 1 \times 4$ rectangular prism such that all four colors appear on each of the four long faces of the prism. The charm of this puzzle lies in the fact that it sounds very easy but is, in fact, quite difficult. Each cube may be given 24 different orientations, and thus there are 82,944 possible rectangular prisms which can be arranged from them (not counting permutation of the order of the cubes or rotation of the prism about its long axis). As we shall see in a moment, only two of these arrangements satisfy the conditions of the puzzle.

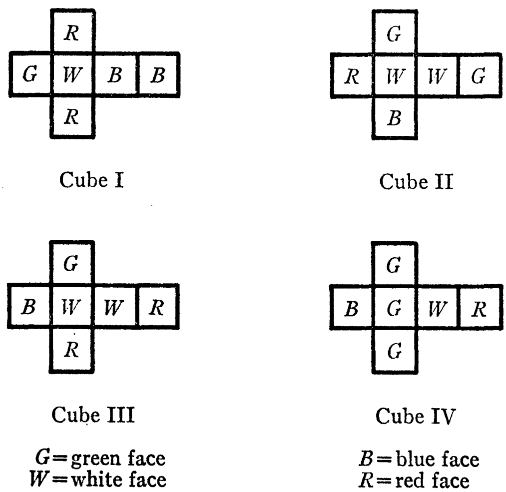


FIG. 1

Some people will (after manipulating the cubes for an hour or so) give up or come to the conclusion that the puzzle is actually impossible to solve. The purpose of this note is to present a method of analyzing puzzles of this type which will permit the solution of this particular puzzle in a minute or two, and which can be used to design similar puzzles which are easier, harder, or actually impossible.

The first fact to note is that, for purposes of this type of puzzle, any colored cube may be replaced by its mirror image (in our particular example Cubes I and IV are their own mirror images, but II and III are not). To see this, imagine that we have assembled the cubes in a rectangular prism on the table in front of us; notice that four faces of each cube appear on long faces of the prism, while two faces of each cube are "buried" or constitute one of the two square faces of the prism (and thus are ignored in determining whether the prism is or is not a solution to the puzzle). If one of the cubes (say II) is replaced by its mirror image (call it II′) then it is obvious that II′ can be rotated in such a way that the four of its faces which appear on long faces of the prism are exactly the same colors in exactly the same positions as before, while its "buried" faces interchange colors. Thus we may characterize a cube (from the standpoint of the puzzle) by naming the three color-pairs appearing on opposite faces. We may set up a correspondence between a subset of the natural numbers and sets of sets of n colored objects in the following way:

Let 1 stand for red.	Let 3 stand for blue.
Let 2 stand for white.	Let 5 stand for green.

Then we let the "characteristic number" of a set of n colored objects be simply the product of the n integers corresponding to their individual colors. For example, the characteristic integer of two blue objects, a red one, and a green one would be $3^2 \cdot 1 \cdot 5 = 45$. Provided we know the number of objects in the set, knowledge of its characteristic number obviously tells us its distribution of colors. Using this code, we easily prepare a list of the characteristic numbers of pairs of opposite faces of each cube:

	(1)		(2)		(3)
Cube I	1	—	6	—	15
Cube II	2	—	10	—	15
Cube III	2	—	5	—	6
Cube IV	5	—	6	—	25

We wish to select a set of four pairs of opposite faces (one pair from each cube) which will constitute the "top" and "bottom" of our prism. Since each color must be represented once on the top and once on the bottom, we want the characteristic number of the total set of eight faces to be $1^2 \cdot 2^2 \cdot 3^2 \cdot 5^2 = 900$. Thus we must select a set of four numbers, one from each row in the 4×3 matrix above, whose product is 900. We must select two such sets: one to determine the "top and bottom" and one to determine the "front and back." The two sets must be disjoint, of course. It is not hard to carry out this task by inspection; however, if you wish to be systematic, it is useful to prepare a list of the nine possible products from cubes I and II, and compare it to a list of the nine possible products from cubes III and IV, as follows:

I II			III IV
(1)×(1)	2	(10)	(1)×(1)
(1)×(2)	10	12	(1)×(2)
(1)×(3)	15	50	(1)×(3)
(2)×(1)	12	25	(2)×(1)
(2)×(2)	60	(30)	(2)×(2)
(2)×(3)	(90)	125	(2)×(3)
(3)×(1)	(30)	30	(3)×(1)
(3)×(2)	150	36	(3)×(2)
(3)×(3)	225	150	(3)×(3)

The only two numbers in the first column which have a multiplicative complement with respect to 900 in the second column are 90 (whose multiplicative complement with respect to 900 is 10) and 30 (whose multiplicative complement with respect to 900 is 30). It is now easy to deduce the following solution to the puzzle:

Cube	"Top and Bottom"	"Front and Back"
I	white—blue	green—blue
II	blue—green	white—red
III	red—white	red—green
IV	green—red	blue—white

The calculations above show that this solution is unique (up to rotation of the prism about its long axis) except for the fact that the colors on the top and bottom may be interchanged without altering the colors on the front and back. Thus there are only two essentially different solutions to the puzzle.

The reader may enjoy using the methods of this paper to solve the following three puzzles, which are presented in increasing order of difficulty.

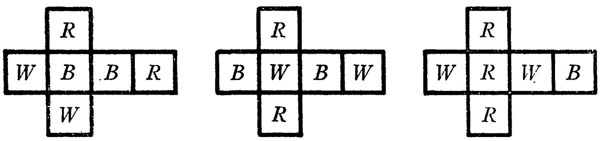
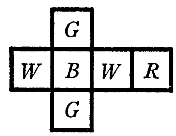
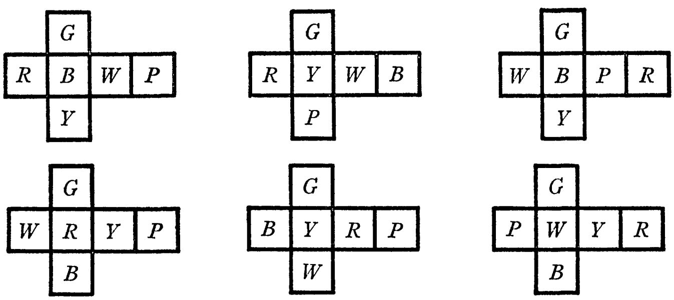


FIG. 2



Cube IV*

FIG. 3



Y=yellow face P=purple face

FIG. 4

1. Determine whether or not it is possible to arrange the three cubes of Figure 2 in a $1\times1\times3$ prism such that the colors red, white, and blue appear on each of the four long faces.

2. Show that the "Instant Insanity" puzzle has no solution if cube IV is replaced by cube IV* (shown in Figure 3).

3. Find a way to arrange the six cubes shown in Figure 4 in a $1\times1\times6$ prism such that all six colors will appear on each long face of the prism. Note that every color appears on every cube; this implies that if there is one solution there must be at least six essentially different solutions. Why?

Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the Rand Corporation or the official opinion or policy of any of its governmental or private research sponsors.

THE SHORTEST CONNECTED GRAPH THROUGH DYNAMIC PROGRAMMING TECHNIQUE

R. C. GARG and SANTOSH KUMAR, Directorate of Scientific Evaluation,
Ministry of Defense, India

1. Introduction. A classical result in network theory is that if a finite connected graph has a positive real number attached to each edge and if these lengths are distinct, then among the spanning trees of a graph there exists a unique graph in which the sum of the edges is a minimum. Kruskal [1] suggested a few constructions for finding this graph under certain restrictions mentioned in his paper.

In the present paper, the problem of interconnecting a given set of terminals with the shortest possible network of direct links has been formulated and solved through the functional equation technique of dynamic programming. Also the restriction imposed by Kruskal of having each edge of distinct length is removed. Some numerical examples are also given to illustrate the process of computation.

The following terms which will be frequently used in the paper are defined below:

ISOLATED TERMINAL. An isolated terminal is a terminal to which, at a given stage of construction, no connection has been made.

FRAGMENT. A fragment is a terminal subset connected by direct links between members of the subset.

CONNECTED GRAPH. A set of terminals is said to be connected if and only if there is an unbroken chain of links between every two terminals of the set.

COMPLETE. A graph is said to be complete when every pair of vertices is connected by the edge. (Any given connected graph, G, can, with no loss in generality, be considered complete. For if any edge of G is missing, an edge of great length may be inserted, and this does not alter the graph which is relevant to the present purpose. Also, it is possible and intuitively appealing to think of missing edges as edges of infinite length.)

2. Direct enumeration method of solving the problem. It is clear from the definition of a complete graph that a network with N terminals will have $N(N-1)/2$ links, while the shortest spanning subtree will have $(N-1)$ links. A direct enumeration will be to formulate $N(N-1)/2\ C_{N-1}$ combinations and choose that minimum which does not contain loops. As N increases, the solution of the problem by direct methods becomes unmanageable. It will be seen that if we solve this problem through dynamic programming technique, the solution is easily obtainable.

3. Dynamic programming formulation of the problem. As discussed above, the shortest connected graph of N terminals will contain $(N-1)$ links. These $(N-1)$ links can be viewed as $(N-1)$ stages of the multistage problem. We define:

f_r = Minimum sum of r links when $(r+1)$ terminals are joined in an optimal way so that r links form a shortest connected graph for $(r+1)$ terminals when $r \geqq 1$ and with $f_0 = 0$.

To obtain the recurrence relation connecting f_r and f_{r-1} for some arbitrary r, we proceed as follows: Let l_r be the length of the rth link. Then, regardless of the precise value of l_r, we know that the remaining $(r-1)$ links will be connected so as to obtain a minimum sum of $(r-1)$ lengths. Since this is also a minimum, then by the definition the sum of $(r-1)$ links will be f_{r-1}. Hence the sum of the r links will be (l_r+f_{r-1}). Now if l_r be chosen in an optimal way, we will have

$$f_r = \min_{l_r} (l_r + f_{r-1})$$

with $f_1=\min (d_{ij})$ where d_{ij} is the length of edge joining the terminal i to terminal j. The starting point i can be taken as any of the given terminals; i.e., take $i=1, 2, \cdots$, or N and obtain a fragment. The decision regarding the minimum l_r shall be to connect the fragment to the nearest neighbor by the smallest link which does not form a loop.

4. **Solution.** Prepare a table for values of d_{ij}. This will be an N by N matrix and will be used to obtain f_2, f_3, etc. We shall continue this process till we obtain f_{N-1}, which is the required shortest connected graph.

We shall now take some numerical examples to illustrate the iterative process of computation.

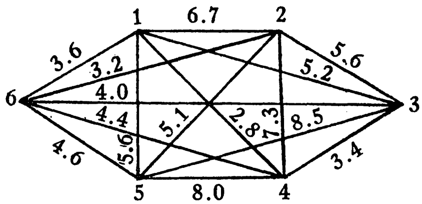


FIG. 1

Example 1. Obtain the shortest connected graph of Figure 1. The values of d_{ij} or f_1, are given in the following table.

TABLE 1

	1	2	3	4	5	6
1	—	6.7	5.2	2.8	5.6	3.6
2	6.7	—	5.7	7.3	5.1	3.2
3	5.2	5.7	—	3.4	8.5	4.0
4	2.8	7.3	3.4	—	8.0	4.4
5	5.6	5.1	8.5	8.0	—	4.6
6	3.6	3.2	4.0	4.4	4.6	—

Let us take terminal 1 as the isolated terminal. Using Table 1 we have the functional $f_1=2.8$ (1–4). The figures in the bracket give the details of the fragment which is obtained by joining terminal 1 to 4.

$$f_2 = \min_{l_2} \left\{ \begin{array}{ll} l_2 + f_1 & \\ 3.6 + 2.8 = 6.4 & \left\{ \begin{array}{l} 6 \\ | \\ 1—4 \end{array} \right\} \\ 3.4 + 2.8 = 6.2 & \left\{ 1—4—3 \right\} \end{array} \right\}$$

$$= 6.2(1—4—3)$$

$$f_3 = \min_{l_3} \left\{ \begin{array}{ll} l_3 + f_2 & \\ 3.6 + 6.2 = 9.8 & \left\{ \begin{array}{l} 6 \\ | \\ 1—4—3 \end{array} \right\} \\ 4.4 + 6.2 = 10.6 & \left\{ \begin{array}{l} 1—4—3 \\ | \\ 6 \end{array} \right\} \\ 4.0 + 6.2 = 10.2 & \left\{ 1—4—3—6 \right\} \end{array} \right\}$$

$$= 9.8 \left\{ \begin{array}{l} 1—4—3 \\ | \\ 6 \end{array} \right\}$$

Similarly we have

$$f_4 = 13.0 \left\{ \begin{array}{l} 1—4—3 \\ | \\ 6—2 \end{array} \right\}$$

and

$$f_5 = 17.6 \left\{ \begin{array}{l} 2—6—1—4—3 \\ \quad\;\; | \\ \quad\;\; 5 \end{array} \right\}.$$

The required shortest connected graph is given in Figure 2.

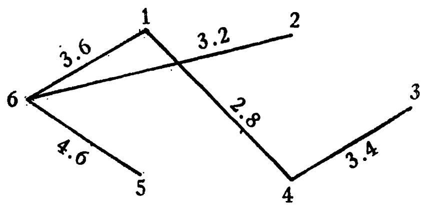


FIG. 2

Example 2. Consider again the graph given in Figure 1 and start from terminal 6. We shall have the following results:

$$f_1 = 3.2(6—2)$$

$$f_2 = 6.8(1—6—2)$$

$$f_3 = 9.6 \left\{ \begin{array}{l} 6—2 \\ | \\ 1—4 \end{array} \right\}$$

$$f_4 = 13.0 \left\{ \begin{array}{l} 6—2 \\ | \\ 1—4—3 \end{array} \right\}$$

$$f_5 = 17.6 \left\{ \begin{array}{l} 2—6—1—4—3 \\ \quad\;\; | \\ \quad\;\; 5 \end{array} \right\}$$

This example indicates that the shortest connected graph is independent of the initial starting point and is unique for a given graph.

Example 3. Obtain the shortest connected graph of Figure 3 (which is not a complete graph and the link lengths are also not distinct).

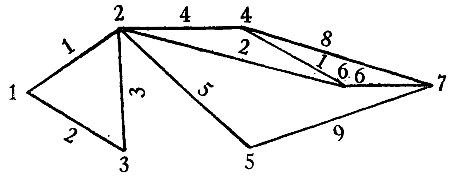


FIG. 3

Taking the starting point as terminal 1, we have the following results:

$$f_1 = 1(1—2)$$

$$f_2 = 3\left\{\begin{array}{l}1—2 \\ | \\ 3\end{array} \quad \text{or} \quad \begin{array}{r}1—2 \\ | \\ 6\end{array}\right\}$$

$$f_3 = 4(1—2—6—4)$$

$$f_4 = 6\left\{\begin{array}{l}1—2—6—4 \\ | \\ 3\end{array}\right\}$$

$$f_5 = 11\left\{\begin{array}{l}1—2—6—4 \\ |\quad\ | \\ 3\quad\ 5\end{array}\right\}$$

$$f_6 = 17\left\{\begin{array}{l}1—2—6—4 \\ |\quad\ |\quad\ | \\ 3\quad\ 5\quad\ 7\end{array}\right\}$$

The shortest connected graph of Figure 3 is shown in Figure 4.

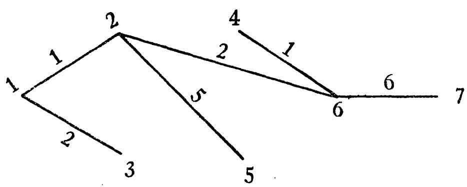


FIG. 4

Acknowledgements. The authors are most grateful to Dr. S. S. Srivastava, Director, Scientfic Evaluation Group, New Delhi; to Dr. R. S. Varma, Dean of the faculty of Mathematics, University of Delhi, Delhi, for their keen interest and guidance in the preparation of this paper. Shri Santosh Kumar is also thankful to Brig. Amar Bakhshi, Director of Armaments and Col. K. S. Tewari, Deputy Director of Armaments for their kind permission to publish this paper.

References

1. J. B. Kruskal, Jr., On the shortest spanning subtree of a graph and the travelling salesman problem, Proc. Amer. Math. Soc., 7 (1956) 48–50.

2. R. Bellman and S. E. Dreyfus, Applied Dynamic Programming, The Rand Corporation Report R-352, PR, 1962.

ON SOLUTIONS OF THE EQUATION $x^a+y^b=z^c$

ALLAN I. LIFF, Lockheed Missiles and Space Company, Sunnyvale, California

Although the solution to Fermat's last theorem is not known, a sufficient condition so that the more general equation

$$x^a + y^b = z^c \tag{1}$$

has integer solutions is that at least one of the known exponents is relatively prime to the other two exponents. Moreover if this condition is met, then there exists at least one family of solutions.

Let us assume that

$$x = p^{ct}q^{bs}r^{bc}, \qquad y = p^n q^{as} r^{ac}, \qquad z = p^{at}q^m r^{ab}, \tag{2}$$

where all the symbols represent positive integers. These equations will be manipulated so that we will obtain a four parameter solution with m, s, q, and r as the parameters. (The main line of reasoning can be seen when $r=1$.) Substituting (2) into (1) and dividing by x^a gives

$$p^{bn-act} + 1 = q^{cm-abs}. \tag{3}$$

When we can determine an n and a t (or alternatively an m and an s) which will make either exponent in (3) unity, we can eliminate either p (or q) from (2) to obtain one family of solutions with the desired integer properties. It is known that the Diophantine equation

$$bn - (ac)t = 1 \tag{4}$$

has positive integer solutions if b and ac are relatively prime ([1], [2]). Assuming that this is the case, we may obtain

$$x = (q^{cm-abs} - 1)^{ct}q^{bs}r^{bc}, \quad y = (q^{cm-abs} - 1)^n q^{as}r^{ac}, \quad z = (q^{cm-abs} - 1)^{at}q^m r^{ab}, \tag{5}$$

where n and t are positive integral solutions of (4) and m, q, s, and r are parameters of the family of solutions. Similarly, we can obtain another family of solutions by solving

$$cm - abs = 1. \tag{6}$$

At this time we can relax the condition that q and r of (5) are integers; it is sufficient that they be rational and so related that x, y, and z are integers.

As an example, let us consider the particular case

$$x^3 + y^4 = z^5 \tag{7}$$

which has recently appeared in the literature ([3], [4]). Using the indicated method in various ways, we may obtain the three families of solutions

$$x = (q^{5m-12s} - 1)^5 q^{4s}r^{20}, \quad y = (q^{5m-12s} - 1)^4 q^{3s}r^{15}, \quad z = (q^{5m-12s} - 1)^3 q^m r^{12}; \tag{8}$$

$$x = (q^{5m-12s} - 1)^7 q^{4s}r^{20}, \quad y = (q^{5m-12s} - 1)^5 q^{3s}r^{15}, \quad z = (q^{5m-12s} - 1)^4 q^m r^{12}; \tag{9}$$

and

$$x = (q^{4g-15h} + 1)^8 q^{5h}r^{20}, \quad y = (q^{4g-15h} + 1)^6 q^g r^{15}, \quad z = (q^{4g-15h} + 1)^5 q^{3h}r^{12}. \tag{10}$$

An apparent fourth solution

(11) $x = (q^{3h-20g} + 1)^8 q^h r^{20}, \quad y = (q^{3h-20g} + 1)^6 q^{5g} r^{15}, \quad z = (q^{3h-20g} + 1)^5 q^{4g} r^{12}$

can be seen to be just a version of (10) by making the substitutions

(12) $q = 1/p, \quad h = 20A - 32B + 115C, \quad g = 3A - 5B + 18C, \quad r = rp^A$

in (11) to obtain (10). Unfortunately the author does not know of any simple logical way to obtain (12). It was obtained by noting the similarity of (10) and (11), applying intuition (insight?), and undetermined coefficient techniques. He also believes that (9), (10), and (11) are different but does not know how to prove it.

References

1. G. Chrystal, Textbook of Algebra, Dover, New York, 1961, chap. 33, sec. 12.
2. H. S. Hall and S. R. Knight. Higher Algebra, Macmillan, London, 1950, sec. 347.
3. D. L. Silverman, Quicky Problem #403, this MAGAZINE, 40, (1967) 54.
4. Litton's Problematical Recreations #366, Electronic News, 12, 587, Feb. 6, 1967, p. 2. Also Aviation Week & Space Technology, 86, #7, Feb. 13, 1967, p. 102. Also Problematical Recreations[9], 29, Litton, 1967.

ON THE VOLUME OF A CLASS OF TRUNCATED PRISMS AND SOME RELATED CENTROID PROBLEMS

MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

The volume of a truncated triangular prism is given by the simple symmetric formula $V=A(e_1+e_2+e_3)/3$ where A is the area of a right section and e_1, e_2, e_3 are the lengths of the lateral edges. We will show that the analogous formula $V=A(e_1+e_2+\cdots+e_n)/n$ for a truncated polygonal prism is valid, if and only if, the centroid of area of any cross-section coincides with the centroid of its vertices (equally weighted). This coincidence of centroids for a quadrilateral implies that it is a parallelogram. By also considering the centroid of the boundary edges of a cross-section, we are led to other problems concerning the coincidence of some or all of the three kinds of centroids.

A truncated prism is the portion of a prism included between one of the bases and a plane not parallel to the base cutting all the edges.

The volume formula for a truncated triangular prism has the simple symmetric form

$$V = A\left(\frac{e_1 + e_2 + e_3}{3}\right) \tag{1}$$

where A is the area of a right section and e_1, e_2, e_3 are the lengths of the lateral edges [1, p. 81]. A volume formula for any truncated polygonal prism can be obtained by dividing up the given truncated prism into truncated triangular prisms and using (1). This does not lead, however, to a simple symmetric expres-

An apparent fourth solution

(11) $x = (q^{3h-20g} + 1)^8 q^h r^{20}, \quad y = (q^{3h-20g} + 1)^6 q^{5g} r^{15}, \quad z = (q^{3h-20g} + 1)^5 q^{4g} r^{12}$

can be seen to be just a version of (10) by making the substitutions

(12) $q = 1/p, \quad h = 20A - 32B + 115C, \quad g = 3A - 5B + 18C, \quad r = rp^A$

in (11) to obtain (10). Unfortunately the author does not know of any simple logical way to obtain (12). It was obtained by noting the similarity of (10) and (11), applying intuition (insight?), and undetermined coefficient techniques. He also believes that (9), (10), and (11) are different but does not know how to prove it.

References

1. G. Chrystal, Textbook of Algebra, Dover, New York, 1961, chap. 33, sec. 12.
2. H. S. Hall and S. R. Knight. Higher Algebra, Macmillan, London, 1950, sec. 347.
3. D. L. Silverman, Quicky Problem #403, this MAGAZINE, 40, (1967) 54.
4. Litton's Problematical Recreations #366, Electronic News, 12, 587, Feb. 6, 1967, p. 2. Also Aviation Week & Space Technology, 86, #7, Feb. 13, 1967, p. 102. Also Problematical Recreations[9], 29, Litton, 1967.

ON THE VOLUME OF A CLASS OF TRUNCATED PRISMS AND SOME RELATED CENTROID PROBLEMS

MURRAY S. KLAMKIN, Ford Scientific Laboratory, Dearborn, Michigan

The volume of a truncated triangular prism is given by the simple symmetric formula $V = A(e_1+e_2+e_3)/3$ where A is the area of a right section and e_1, e_2, e_3 are the lengths of the lateral edges. We will show that the analogous formula $V = A(e_1+e_2+\cdots+e_n)/n$ for a truncated polygonal prism is valid, if and only if, the centroid of area of any cross-section coincides with the centroid of its vertices (equally weighted). This coincidence of centroids for a quadrilateral implies that it is a parallelogram. By also considering the centroid of the boundary edges of a cross-section, we are led to other problems concerning the coincidence of some or all of the three kinds of centroids.

A truncated prism is the portion of a prism included between one of the bases and a plane not parallel to the base cutting all the edges.

The volume formula for a truncated triangular prism has the simple symmetric form

(1) $$V = A\left(\frac{e_1 + e_2 + e_3}{3}\right)$$

where A is the area of a right section and e_1, e_2, e_3 are the lengths of the lateral edges [1, p. 81]. A volume formula for any truncated polygonal prism can be obtained by dividing up the given truncated prism into truncated triangular prisms and using (1). This does not lead, however, to a simple symmetric expression as (1). But it does lead one to consider the problem of characterizing those polygonal prisms such that the volumes of *all* their truncations have a volume formula analogous to (1), i.e.,

THEOREM 1.

$$V = A \left\{ \frac{e_1 + e_2 + \cdots e_n}{n} \right\} \tag{2}$$

if the area centroid of the base coincides with the centroid of the vertices.

Proof. By constructing a right cross-section of the prism, the volume of the prism can be obtained as the sum or difference of two truncated right prisms with the same base. (See Figure 1.) (By a right truncated prism, we mean one in which there is a base perpendicular to all the edges.) Consequently, we can assume without loss of generality that our prisms are right ones.

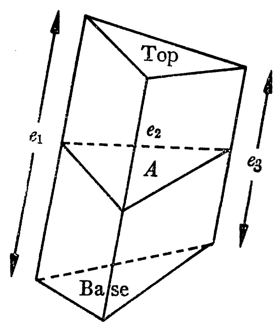


FIG. 1

If we take the base B to lie in the (x, y) plane and take the equation of the truncating plane to be $z = ax + by + c$, the volume of the truncated prism is then given by

$$V = \int_B \int z\,dx\,dy.$$

If, in addition, we take the origin of our coordinate system to coincide with the area centroid of B, the above expression simplifies to

$$V = \int_B \int c\,dx\,dy = cA,$$

where A denotes the area of the base B.

Since area centroids (and also centroids of the vertices, each being considered equally weighted) project into corresponding centroids under parallel projection, c also is the height of the centroid of the truncating cross-section above the base. In Figure 1, this will imply that the three centroids of area (or of the vertices) of the three sections, top, A, and the base, lie on a line parallel to the edges. Also, since $(e_1 + e_2 + \cdots + e_n)/n$ equals the height of the centroid of the vertices (considered equally weighted) of the truncating cross-section above the base, (2) will be valid for *all* truncations, if and only if, the area centroid of the base coincides with the centroid of its vertices.

Since the above two types of centroids coincide for any triangle, (1) holds for any truncated triangular prism. This provides an alternate proof to the one referred to before. For quadrilaterals it will be shown that the two kinds of centroids coincide, if and only if, the quadrilaterals are parallelograms. (This result is not new. It is given as an exercise in [2, p. 81], and is credited to a Cambridge college examination paper. However, the proofs given here may be new.) For higher order polygons, there are not such restrictive results and this is to be expected since there are more degrees of freedom. However, a simple sufficient condition for the two kinds of centroids to coincide is that the polygon be either centrosymmetric or else the parallel projection of an odd regular polygon.

The coincidence of the two kinds of centroids of any triangle follows easily by direct calculation since they both must be at the intersection of the medians. It is of interest to note that the result also follows immediately from the known result that any triangle can be parallelly projected into an equilateral triangle in which case the result is immediate by symmetry.

To show that the only quadrilaterals in which the two centroids coincide are parallelograms, we will use the following elegant result as a lemma. (Proved in [2], p. 162.)

LEMMA 1. *The area centroid of a quadrilateral coincides with that of four particles of equal mass placed at the four corners together with a fifth particle of equal but negative mass placed at the intersections of the diagonals.*

For completeness here, we now include the proof.

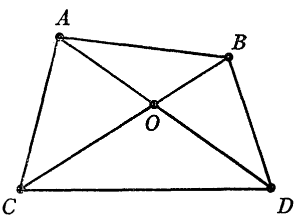


FIG. 2

Proof. Let $3m$ and $3n$ denote the areas of ΔABC and ΔCBD, respectively. (See Figure 2.) By the previous result on triangles, we may replace ΔABC by particles of mass m at each point A, B, C and replace ΔCBD by particles of mass n at each point C, B, D. Furthermore, we may add particles of mass n at A and m at D, so as to make the four particles at A, B, C, D have the same mass $m+n$, provided we counterbalance the added masses by placing mass $-(m+n)$ at the centroid of mass n at A and mass m at D. But this latter centroid is at 0 since

$$\frac{\text{Area } \Delta ABC}{\text{Area } \Delta CBD} = \frac{m}{n} = \frac{AO}{OD}.$$

This establishes our first lemma. Now note that the centroid of the four vertices A, B, C, D is at the midpoint of either segment joining the midpoints of AB and CD or AC and BD. The proof for the quadrilateral being a parallelogram is now an immediate corollary of

THEOREM 2. *If the point of intersection of the diagonals of a quadrilateral $ABCD$ lies on the segment joining the midpoints of AB and CD, then AB is parallel to CD.*

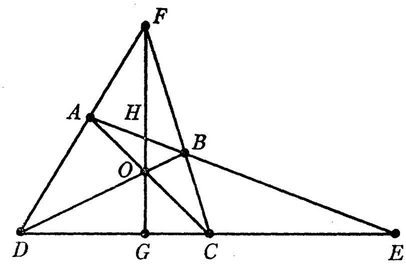


FIG. 3

Proof. We prove the theorem by contradiction. Assume AB intersects CD and consider the complete quadrilateral as shown in Figure 3. It is a known result that points $\{DGCE\}$ and $\{AHBE\}$ form harmonic ranges and that the midpoints of DC and AB, respectively, lie to the left of line HG. ($\{DGCE\}$ is a harmonic range if $DG/GC = DE/CE$. See [3], pp. 65, 82.) This gives us a contradiction since the centroid of A, B, C, D would then be off the line HG. Consequently, AB is parallel to CD.

COROLLARY 1. *If the area centroid of a quadrilateral coincides with the centroid of its vertices, then the quadrilateral is a parallelogram.*

Proof. This follows immediately from the note at the end of our first lemma and the preceding theorem. An alternate proof of Corollary 1 using Lemma 1, follows by noting that the centroid of the vertices A, B, C, D must be at the point of intersection, O, of the diagonals. Thus, $\mathbf{OA}+\mathbf{OB}+\mathbf{OC}+\mathbf{OD}=\mathbf{0}$ which implies that $\mathbf{OA}+\mathbf{OD}=\mathbf{0}=\mathbf{OB}+\mathbf{OC}$.

The previous results on the coincidence of the centroids for triangles and quadrilaterals can also be established analytically by using the following vector formulation:

THEOREM 3. *If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ denote coterminal vectors for a nondegenerate triangle (as in Figure 4), then $(P)\Leftrightarrow(Q)$ where*

$$(P):\ \mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{0},$$
$$(Q):\ (\mathbf{A}+\mathbf{B})|\mathbf{A}\times\mathbf{B}| + (\mathbf{B}+\mathbf{C})|\mathbf{B}\times\mathbf{C}| + (\mathbf{C}+\mathbf{A})|\mathbf{C}\times\mathbf{A}| = \mathbf{0}.$$

Note that statement (P) is equivalent to the statement that the centroid of the vertices (i.e., $\{\mathbf{A}+\mathbf{B}+\mathbf{C}\}/3$) is at point O. Also, statement (Q) is equivalent to the statement that the area centroid of $\triangle ABC$ is at point O since $|\mathbf{A}\times\mathbf{B}|$ is twice the area of $\triangle OAB$ whose area centroid is given by $\{\mathbf{A}+\mathbf{B}\}/3$.

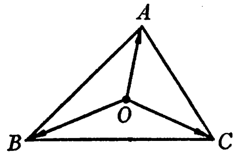


FIG. 4

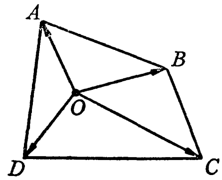


FIG. 5

Proof. 1. If (P) holds, then by multiplying by $\boldsymbol{A}\times$ and $\boldsymbol{B}\times$, we obtain

$$|\boldsymbol{A}\times\boldsymbol{B}| = |\boldsymbol{B}\times\boldsymbol{C}| = |\boldsymbol{C}\times\boldsymbol{A}|.$$

Thus (Q) reduces to $2(\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C})|\boldsymbol{A}\times\boldsymbol{B}|=\boldsymbol{0}$.

2. If (Q) holds, then $\boldsymbol{C}=m\boldsymbol{A}+n\boldsymbol{B}$ where m and n are negative scalar quantities. On substituting back in (Q), we obtain

$$\{\boldsymbol{A}+\boldsymbol{B}\} - m\{m\boldsymbol{A}+(n+1)\boldsymbol{B}\} - n\{(m+1)\boldsymbol{A}+n\boldsymbol{B}\} = \boldsymbol{0}.$$

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are linearly independent, m and n must satisfy $1-m^2-m(n+1)=0$ and $1-n^2-n(m+1)=0$. Since m and n are negative, it follows by subtraction that $m=n=-1$.

LEMMA 2. *If the centroid O of the vertices A, B, C, D, of a quadrilateral divides the figure into triangles such that*

$$\text{Area } \Delta AOB = \text{Area } \Delta DOC,$$

$$\text{Area } \Delta AOD = \text{Area } \Delta BOC,$$

then the area centroid of the quadrilateral coincides with O and $ABCD$ is a parallelogram.

Proof. (See Figure 5.) $\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\boldsymbol{D}=\boldsymbol{0}$ since we have chosen the origin of the four vectors to be at O. By equality of areas, $\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{C}\times\boldsymbol{D}$, $\boldsymbol{C}\times\boldsymbol{B}=\boldsymbol{A}\times D$. By adding, $(\boldsymbol{A}+\boldsymbol{C})\times(\boldsymbol{B}-\boldsymbol{D})=\boldsymbol{0}$ or $\boldsymbol{0}=(\boldsymbol{B}+\boldsymbol{D})\times(\boldsymbol{B}-\boldsymbol{D})=\boldsymbol{B}\times\boldsymbol{D}$. Thus $\boldsymbol{B}\|\boldsymbol{D}$ and, similarly, $\boldsymbol{A}\|\boldsymbol{C}$ which implies that $\boldsymbol{A}+\boldsymbol{C}=\boldsymbol{0}=\boldsymbol{B}+\boldsymbol{D}$ or that the figure is a parallelogram.

As a converse of Lemma 2, we have

COROLLARY 2. *If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ denote coterminal vectors for a nondegenerate quadrilateral (as in Figure 5) where*

$$\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}+\boldsymbol{D}=\boldsymbol{0} \tag{3}$$

and

$$(\boldsymbol{A}+\boldsymbol{B})|\boldsymbol{A}\times\boldsymbol{B}| + (\boldsymbol{B}+\boldsymbol{C})|\boldsymbol{B}\times\boldsymbol{C}| + (\boldsymbol{C}+\boldsymbol{D})|\boldsymbol{C}\times\boldsymbol{D}| + (\boldsymbol{D}+\boldsymbol{A})|\boldsymbol{D}\times\boldsymbol{A}| = \boldsymbol{0}, \tag{4}$$

then $ABCD$ is a parallelogram and also $\boldsymbol{A}+\boldsymbol{C}=\boldsymbol{0}=\boldsymbol{B}+\boldsymbol{D}$.

Proof. (3) and (4) imply

$$(5)\quad (\boldsymbol{A}+\boldsymbol{B})\{|\boldsymbol{A}\times\boldsymbol{B}|-|\boldsymbol{C}\times\boldsymbol{D}|\}+(\boldsymbol{B}+\boldsymbol{C})\{|\boldsymbol{B}\times\boldsymbol{C}|-|\boldsymbol{D}\times\boldsymbol{A}|\}=0,$$

$$(6)\quad (\boldsymbol{A}+\boldsymbol{B})\{|\boldsymbol{A}\times\boldsymbol{B}|-|\boldsymbol{C}\times\boldsymbol{D}|\}+(\boldsymbol{D}+\boldsymbol{A})\{|\boldsymbol{D}\times\boldsymbol{A}|-|\boldsymbol{B}\times\boldsymbol{C}|\}=0.$$

Since $\boldsymbol{C}+\boldsymbol{D}$, $\boldsymbol{B}+\boldsymbol{C}$, and $\boldsymbol{D}+\boldsymbol{A}$ cannot all be nontrivially parallel to $\boldsymbol{A}+\boldsymbol{B}$, the scalar coefficients of (5) and (6) must vanish. Thus,

$$(7)\qquad |\boldsymbol{A}\times\boldsymbol{B}|=|\boldsymbol{C}\times\boldsymbol{D}| \quad\text{and}\quad |\boldsymbol{B}\times\boldsymbol{C}|=|\boldsymbol{D}\times\boldsymbol{A}|.$$

Since (7) implies the equality of areas of the pairs of triangles AOB with DOC and AOD with BOC, Lemma 2 now applies.

Another way, but much more troublesome, of establishing Theorem 4 would be to set

$$\boldsymbol{C}=m\boldsymbol{A}+n\boldsymbol{B} \quad\text{and}\quad \boldsymbol{D}=r\boldsymbol{A}+s\boldsymbol{B},$$

where m, n, r and s are scalars such that (3) and (4) are satisfied. This leads to the following four nonlinear simultaneous equations:

$$m+r+1=0,$$
$$n+s+1=0,$$
$$(m+r)|ms-nr|+(r+1)|s|+m|m|+1=0,$$
$$(n+s)|ms-nr|+(n+1)|m|+s|s|+1=0.$$

It follows from the previous argument that the only solution is

$$m=-1,\qquad n=0,\qquad r=0,\qquad s=-1.$$

We now consider some analogous problems relating to a third centroid of the polygon, the one of the sides. For convenience, we will use the notation C_v^n, C_a^n, C_s^n to denote the centroids of the vertices, of the area, and the sides, respectively, of an n-gon. $C_v^4\cong C_s^4$ will denote that the centroid of a quadrilateral with respect to area is the same as the centroid with respect to the sides.

The only new problem with respect to triangles, is to determine all triangles such that $C_v^3\cong C_s^3$.

We cannot use parallel projection here since ratio of lengths is not preserved in general. (It is, however, for parallel segments.)

Analytically, we have

$$(8)\qquad \boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}=0,$$

$$(9)\qquad (\boldsymbol{A}+\boldsymbol{B})|\boldsymbol{A}-\boldsymbol{B}|+(\boldsymbol{B}+\boldsymbol{C})|\boldsymbol{B}-\boldsymbol{C}|+(\boldsymbol{C}+\boldsymbol{A})|\boldsymbol{C}-\boldsymbol{A}|=0.$$

Using (8), (9) reduces to

$$(10)\qquad -\boldsymbol{C}=\frac{\boldsymbol{A}|\boldsymbol{B}-\boldsymbol{C}|+\boldsymbol{B}|\boldsymbol{C}-\boldsymbol{A}|}{|\boldsymbol{A}-\boldsymbol{B}|}.$$

Comparing (8) and (10), we get

$$|\boldsymbol{A}-\boldsymbol{B}|=|\boldsymbol{B}-\boldsymbol{C}|=|\boldsymbol{C}-\boldsymbol{A}|$$

which implies that the triangle is equilateral.

For quadrilaterals, we have two new problems, i.e.,

$$C_v^4 \cong C_s^4, \tag{11}$$

$$C_a^4 \cong C_s^4. \tag{12}$$

For (11), we proceed in the same way as for $C_v^3 \cong C_s^3$ (see Corollary 2). This leads to

$$|A - B| = |C - D| \quad \text{and} \quad |B - C| = |D - A|,$$

and thus the figure is again a parallelogram.

The equations for (12) are

$$(A+B)\,|A\times B| + (B+C)\,|B\times C| + (C+D)\,|C\times D| + (D+A)\,|D\times A| = 0,$$

$$(A+B)\,|A-B| + (B+C)\,|B-C| + (C+D)\,|C-D| + (D+A)\,|D-A| = 0.$$

It seems reasonable to conjecture that (12) implies $A+B+C+D=0$ and, consequently, that the quadrilateral is a parallelogram.

Another open question is whether or not $C_v^5 \cong C_a^5 \cong C_s^5$ implies that the pentagon is regular.

References

1. W. F. Kern and J. R. Bland, Solid Mensuration with Proofs, Wiley, New York, 1938.
2. A. S. Ramsey, Statics, Cambridge University Press, Cambridge, 1949.
3. C. V. Durell, Modern Geometry, Macmillan, London, 1952.

ON PROPERTIES PRESERVED BY CONTINUOUS FUNCTIONS

MICHAEL GEMIGNANI, SUNY at Buffalo and Smith College

Generally, if a particular property is invariant under continuous functions, then this invariance is proved for the particular property on an individual basis, e.g., there are separate proofs for compactness and connectedness. It would be valuable, however, to have a general characterization of all of the properties preserved by continuous functions, or to know at least that if a property fell into some large and "natural" class of properties, then it is invariant under continuous functions. This paper is a limited effort in the latter direction.

A *map* shall be any continuous and onto function.

We say that a property P is *regressive* if whenever a topology τ has P, then any topology τ' coarser than or equal to τ (that is, $\tau' \subseteq \tau$) also has P. The properties of being compact, connected, Lindelöf, or countably compact are examples of regressive properties; these properties are also preserved by maps.

If a property is preserved by all maps, then that property must be regressive. For if τ and τ' are both topologies on a set X and $\tau' \subseteq \tau$, then the identity function $i: X, \tau \to X, \tau'$ is continuous and onto; hence any property possessed by τ which is preserved by maps is possessed by τ' as well. It is not true, however, that every

For quadrilaterals, we have two new problems, i.e.,

$$C_v^4 \cong C_s^4, \tag{11}$$

$$C_a^4 \cong C_s^4. \tag{12}$$

For (11), we proceed in the same way as for $C_v^3 \cong C_s^3$ (see Corollary 2). This leads to

$$|A-B| = |C-D| \quad\text{and}\quad |B-C| = |D-A|,$$

and thus the figure is again a parallelogram.

The equations for (12) are

$$(A+B)|A\times B| + (B+C)|B\times C| + (C+D)|C\times D| + (D+A)|D\times A| = 0,$$
$$(A+B)|A-B| + (B+C)|B-C| + (C+D)|C-D| + (D+A)|D-A| = 0.$$

It seems reasonable to conjecture that (12) implies $A+B+C+D=0$ and, consequently, that the quadrilateral is a parallelogram.

Another open question is whether or not $C_v^5 \cong C_a^5 \cong C_s^5$ implies that the pentagon is regular.

References

1. W. F. Kern and J. R. Bland, Solid Mensuration with Proofs, Wiley, New York, 1938.
2. A. S. Ramsey, Statics, Cambridge University Press, Cambridge, 1949.
3. C. V. Durell, Modern Geometry, Macmillan, London, 1952.

ON PROPERTIES PRESERVED BY CONTINUOUS FUNCTIONS

MICHAEL GEMIGNANI, SUNY at Buffalo and Smith College

Generally, if a particular property is invariant under continuous functions, then this invariance is proved for the particular property on an individual basis, e.g., there are separate proofs for compactness and connectedness. It would be valuable, however, to have a general characterization of all of the properties preserved by continuous functions, or to know at least that if a property fell into some large and "natural" class of properties, then it is invariant under continuous functions. This paper is a limited effort in the latter direction.

A *map* shall be any continuous and onto function.

We say that a property P is *regressive* if whenever a topology τ has P, then any topology τ' coarser than or equal to τ (that is, $\tau' \subseteq \tau$) also has P. The properties of being compact, connected, Lindelöf, or countably compact are examples of regressive properties; these properties are also preserved by maps.

If a property is preserved by all maps, then that property must be regressive. For if τ and τ' are both topologies on a set X and $\tau' \subseteq \tau$, then the identity function $i: X, \tau \to X, \tau'$ is continuous and onto; hence any property possessed by τ which is preserved by maps is possessed by τ' as well. It is not true, however, that every regressive property is preserved by maps; for example, the property of not being T_0 is regressive, but is not preserved by maps. Nevertheless, the following proposition indicates that the property of being regressive is "almost sufficient" for invariance under maps.

PROPOSITION 1: *Let P be any regressive property. Then the following statements are equivalent:*

(a) *P is preserved by any map.*

(b) *P is preserved by any open map.*

(c) *P is preserved by any perfect map. (A map $f\colon X\to Y$ is perfect if f is closed (and continuous), and for each y in Y, $f^{-1}(y)$ is compact.)*

(d) *P is preserved by any one-one map.*

(e) *P is preserved by any monotone map. (A map $f\colon X\to Y$ is monotone if for each y in Y, $f^{-1}(y)$ is connected.)*

(f) *If X is any space with a topology τ having P, then the quotient topology for any quotient space of X, τ also has P.*

Proof. Clearly, (a) implies (b) through (e). Suppose $f\colon X, \tau_X \to Y, \tau_Y$. Define $f^{-1}(\tau_Y) = \{U \mid U = f^{-1}(V),\ V \in \tau_Y\}$. Then $f^{-1}(\tau_Y)$ is a topology on X with $f^{-1}(\tau_Y) \subseteq \tau_X$. Moreover, $f\colon X, f^{-1}(\tau_Y) \to Y, \tau_Y$ is easily seen to be open, perfect and monotone, and is one-one if f is one-one. If τ_X has P and P is regressive, then $f^{-1}(\tau_Y)$ has P. Hence if one of the statements (b) through (e) holds, then τ_Y also has P. Consequently, any of the statements (b) through (e) imply (a).

(a) implies (f): One can use the usual identification map from any space onto any of its quotient spaces.

(f) implies (a): Suppose $f\colon X, \tau_X \to Y, \tau_Y$ is a map and τ_X has P. Since P is regressive, $f^{-1}(\tau_Y)$ also has P. Define an equivalence relation R on X by xRx' if $f(x) = f(x')$. If X/R is given the quotient topology from $f^{-1}(\tau_Y)$, then X/R is homeomorphic to Y. But since $f^{-1}(\tau_Y)$ has P, then τ_Y must also have P.

COROLLARY 1. *The following properties are preserved by open maps, but not by every map and hence are not regressive: locally compact, locally connected, paracompact, and first, or second countable.*

COROLLARY 2. *The following properties are preserved by perfect maps, but not by every map and hence are not regressive: T_2, regular, normal, and metrizable and second countable.*

We now show that a certain type of regressive property is always preserved by maps.

If τ is a topology, let $|\tau|$ denote the lattice of τ-open sets (partially ordered by inclusion). A property P of τ is said to be a lattice property if P depends only on $|\tau|$.

PROPOSITION 2. *If P is a regressive lattice property, then P is preserved by any map.*

Proof. We use the situation set up in the proof of (f) implies (a) from Proposision 1. $|f^{-1}(\tau_Y)|$ is isomorphic to the lattice of open sets of the quotient space by the isomorphism defined by $U \to U/R$ for each U of $f^{-1}(\tau_Y)$.

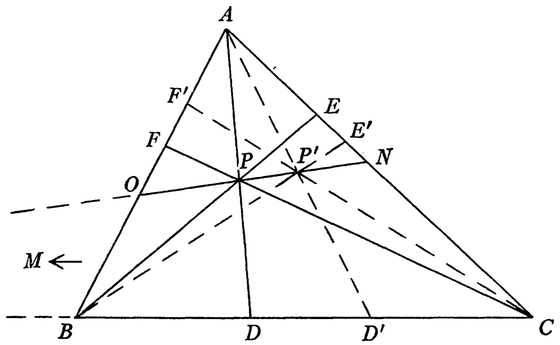
References

1. James Dugundji, Topology, Allyn and Bacon, Boston, 1966.
2. John Hocking and Gail Young, Topology, Addison-Wesley, Reading, 1961.

LINE DETERMINED BY TWO POINTS

D. MOODY BAILEY, Princeton, W. Va.

Suppose that P is any point in the plane of triangle ABC. Construct rays AP, BP, CP to meet sides BC, CA, AB at points D, E, F respectively. Triangle DEF then becomes the cevian triangle of point P with respect to triangle ABC.



Allow any straight line through P to meet CA at N and AB at O. Consider triangle ABE and transversal FPC. The theorem of Menelaus yields $(AF/FB)(BP/PE)(EC/CA)=-1$, or $AF/FB=(EP/PB)(AC/EC)$. Again, triangle ABE and transversal OPN give $(BO/OA)(AN/NE)(EP/PB)=-1$, or $BO/OA=(EN/AN)(PB/EP)$.

By multiplying the values thus secured for AF/FB and BO/OA, it is found that

$$\frac{AF}{FB}\cdot\frac{BO}{OA}=\frac{EP}{PB}\cdot\frac{AC}{EC}\cdot\frac{EN}{AN}\cdot\frac{PB}{EP}=\frac{AC}{EC}\cdot\frac{EN}{AN}=\left(\frac{AE+EC}{EC}\right)\left(\frac{EC-NC}{AN}\right)$$

$$=\left(\frac{AE}{EC}+1\right)\left(\frac{EC}{AN}-\frac{NC}{AN}\right)=\frac{AE}{AN}+\frac{EC}{AN}-\frac{AE}{EC}\cdot\frac{NC}{AN}-\frac{NC}{AN}.$$

Since the preceding computation yields

$$\frac{AF}{FB}\cdot\frac{BO}{OA}=\frac{AE}{AN}+\frac{EC}{AN}-\frac{AE}{EC}\cdot\frac{NC}{AN}-\frac{NC}{AN},$$

it is evident that

$$\frac{AF}{FB}\cdot\frac{BO}{OA}+\frac{AE}{EC}\cdot\frac{NC}{AN}=\frac{AE+EC-NC}{AN}=\frac{AN}{AN}=1\,.$$

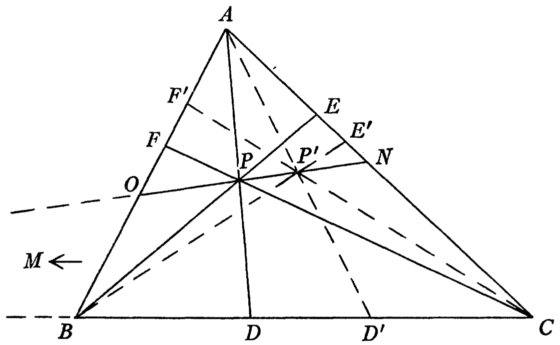
References

1. James Dugundji, Topology, Allyn and Bacon, Boston, 1966.
2. John Hocking and Gail Young, Topology, Addison-Wesley, Reading, 1961.

LINE DETERMINED BY TWO POINTS

D. MOODY BAILEY, Princeton, W. Va.

Suppose that P is any point in the plane of triangle ABC. Construct rays AP, BP, CP to meet sides BC, CA, AB at points D, E, F respectively. Triangle DEF then becomes the cevian triangle of point P with respect to triangle ABC.



Allow any straight line through P to meet CA at N and AB at O. Consider triangle ABE and transversal FPC. The theorem of Menelaus yields $(AF/FB)(BP/PE)(EC/CA)=-1$, or $AF/FB=(EP/PB)(AC/EC)$. Again, triangle ABE and transversal OPN give $(BO/OA)(AN/NE)(EP/PB)=-1$, or $BO/OA=(EN/AN)(PB/EP)$.

By multiplying the values thus secured for AF/FB and BO/OA, it is found that

$$\frac{AF}{FB}\cdot\frac{BO}{OA}=\frac{EP}{PB}\cdot\frac{AC}{EC}\cdot\frac{EN}{AN}\cdot\frac{PB}{EP}=\frac{AC}{EC}\cdot\frac{EN}{AN}=\left(\frac{AE+EC}{EC}\right)\left(\frac{EC-NC}{AN}\right)$$

$$=\left(\frac{AE}{EC}+1\right)\left(\frac{EC}{AN}-\frac{NC}{AN}\right)=\frac{AE}{AN}+\frac{EC}{AN}-\frac{AE}{EC}\cdot\frac{NC}{AN}-\frac{NC}{AN}.$$

Since the preceding computation yields

$$\frac{AF}{FB}\cdot\frac{BO}{OA}=\frac{AE}{AN}+\frac{EC}{AN}-\frac{AE}{EC}\cdot\frac{NC}{AN}-\frac{NC}{AN},$$

it is evident that

$$\frac{AF}{FB}\cdot\frac{BO}{OA}+\frac{AE}{EC}\cdot\frac{NC}{AN}=\frac{AE+EC-NC}{AN}=\frac{AN}{AN}=1\,.$$

By reversing the direction of the two segments in ratio NC/AN, this latter result may be rewritten as

$$\frac{AF}{FB}\cdot\frac{BO}{OA}+\frac{AE}{EC}\cdot\frac{CN}{NA}=1.$$

If line NO be extended to meet BC at point M, we may proceed in like fashion to show that

$$\frac{BD}{DC}\cdot\frac{CM}{MB}+\frac{BF}{FA}\cdot\frac{AO}{OB}=1 \quad\text{and}\quad \frac{CE}{EA}\cdot\frac{AN}{NC}+\frac{CD}{DB}\cdot\frac{BM}{MC}=1.$$

THEOREM 1. *P is any point in the plane of triangle ABC, with DEF its cevian triangle. Any straight line through P meets sides BC, CA, AB at respective points M, N, O so that*

$$\frac{AF}{FB}\cdot\frac{BO}{OA}+\frac{AE}{EC}\cdot\frac{CN}{NA}=1, \qquad \frac{BD}{DC}\cdot\frac{CM}{MB}+\frac{BF}{FA}\cdot\frac{AO}{OB}=1,$$

$$\frac{CE}{EA}\cdot\frac{AN}{NC}+\frac{CD}{DB}\cdot\frac{BM}{MC}=1.$$

Conversely, if any of these equations is satisfied, line MNO must pass through point P.

Allow P and P' to be a pair of points in the plane of triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. Construct line PP' to meet BC at M, CA at N, AB at O. Now line MNO passes through point P and Theorem 1 yields

$$\frac{AF}{FB}\cdot\frac{BO}{OA}+\frac{AE}{EC}\cdot\frac{CN}{NA}=1.$$

By solving this equation for BO/OA, we obtain

$$\text{(1)} \qquad \frac{BO}{OA}=\left(1-\frac{AE}{EC}\cdot\frac{CN}{NA}\right)\frac{BF}{FA}=\frac{BF}{FA}-\frac{BD}{DC}\cdot\frac{CN}{NA}.$$

The reader should observe that BD/DC has replaced $(AE/EC)(BF/FA)$. Ceva's equation $(BD/DC)(CE/EA)(AF/FB)=1$ shows that the two quantities are equivalent.

Line MNO also passes through point P' and Theorem 1 gives

$$\frac{AF'}{F'B}\cdot\frac{BO}{OA}+\frac{AE'}{E'C}\cdot\frac{CN}{NA}=1,$$

or

$$\text{(2)} \qquad \frac{BO}{OA}=\left(1-\frac{AE'}{E'C}\cdot\frac{CN}{NA}\right)\frac{BF'}{F'A}=\frac{BF'}{F'A}-\frac{BD'}{D'C}\cdot\frac{CN}{NA}.$$

As before, $(AE'/E'C)(BF'/F'A)$ is replaced by $BD'/D'C$, since $(BD'/D'C)(CE'/E'A)(AF'/F'B)=1$.

From equations (1) and (2), we have

$$\frac{BF}{FA}-\frac{BD}{DC}\cdot\frac{CN}{NA}=\frac{BF'}{F'A}-\frac{BD'}{D'C}\cdot\frac{CN}{NA}.$$

A solution of this equation shows that

$$\frac{CN}{NA}=\frac{BF/FA-BF'/F'A}{BD/DC-BD'/D'C}.$$

In a similar manner the equations of Theorem 1 may be used to calculate the ratios BM/MC and AO/OB.

THEOREM 2. *P and P' are two points in the plane of triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. A straight line through P and P' meets sides BC, CA, AB at points M, N, O respectively. Under these conditions*

$$\frac{BM}{MC}=\frac{AE/EC-AE'/E'C}{AF/FB-AF'/F'B},\qquad \frac{CN}{NA}=\frac{BF/FA-BF'/F'A}{BD/DC-BD'/D'C},$$
$$\frac{AO}{OB}=\frac{CD/DB-CD'/D'B}{CE/EA-CE'/E'A}.$$

If the cevian ratios associated with any pair of points are known, Theorem 2 enables us to calculate the ratios determined by a line through these points. As an example, let P and P' be the Gergonne point and incenter respectively of triangle ABC. If a, b, c represent sides BC, CA, AB of the given triangle, it is known that

$$\frac{BD}{DC}=\frac{a+c-b}{a+b-c},\qquad \frac{CE}{EA}=\frac{a+b-c}{b+c-a},\qquad \frac{AF}{FB}=\frac{b+c-a}{a+c-b}$$

and

$$\frac{BD'}{D'C}=\frac{c}{b},\qquad \frac{CE'}{E'A}=\frac{a}{c},\qquad \frac{AF'}{F'B}=\frac{b}{a}.$$

By substituting these ratios in the equations of Theorem 2, values are found for BM/MC, CN/NA, AO/OB. In this instance line MNO becomes Soddy's line with respect to triangle ABC. The centers of two of Soddy's circles lie on this line.

THEOREM 2A. *Soddy's line meets the sides of triangle ABC at points M, N, O so that*

$$\frac{BM}{MC}=\left(\frac{c-a}{b-a}\right)\left(\frac{a+c-b}{a+b-c}\right)^2,\qquad \frac{CN}{NA}=\left(\frac{a-b}{c-b}\right)\left(\frac{a+b-c}{b+c-a}\right)^2,$$
$$\frac{AO}{OB}=\left(\frac{b-c}{a-c}\right)\left(\frac{b+c-a}{a+c-b}\right)^2.$$

Let P and P' be the isogonic centers of triangle ABC. The two points then have the property that from either of them the angles subtended by the sides are 60° or 120°. In this case the derivation of ratio values for points P and P' involves a lengthy calculation and for that reason the computation is omitted. However, the author has determined these ratio values and by using them in Theorem 2 has secured the following result:

THEOREM 2B. *A line through the isogonic centers of triangle ABC meets sides BC, CA, AB at points M, N, O so that*

$$\frac{BM}{MC}=\left(\frac{c^2-a^2}{b^2-a^2}\right)\left(\frac{a^2+c^2-b^2-ac}{a^2+b^2-c^2-ab}\right)\left(\frac{a^2+c^2-b^2+ac}{a^2+b^2-c^2+ab}\right),$$

$$\frac{CN}{NA}=\left(\frac{a^2-b^2}{c^2-b^2}\right)\left(\frac{a^2+b^2-c^2-ab}{b^2+c^2-a^2-bc}\right)\left(\frac{a^2+b^2-c^2+ab}{b^2+c^2-a^2+bc}\right),$$

$$\frac{AO}{OB}=\left(\frac{b^2-c^2}{a^2-c^2}\right)\left(\frac{b^2+c^2-a^2-bc}{a^2+c^2-b^2-ac}\right)\left(\frac{b^2+c^2-a^2+bc}{a^2+c^2-b^2+ac}\right).$$

Allow P to be the symmedian point so that

$$\frac{BD}{DC}=\frac{c^2}{b^2},\qquad \frac{CE}{EA}=\frac{a^2}{c^2},\qquad \frac{AF}{FB}=\frac{b^2}{a^2}.$$

If P' be the de Longchamps' point, it can be shown that

$$\frac{BD'}{D'C}=\frac{a^4+b^4-3c^4-2a^2b^2+2b^2c^2+2a^2c^2}{a^4+c^4-3b^4-2a^2c^2+2b^2c^2+2a^2b^2},$$

$$\frac{CE'}{E'A}=\frac{b^4+c^4-3a^4-2b^2c^2+2a^2c^2+2a^2b^2}{a^4+b^4-3c^4-2a^2b^2+2a^2c^2+2b^2c^2}.$$

$$\frac{AF'}{F'B}=\frac{a^4+c^4-3b^4-2a^2c^2+2a^2b^2+2b^2c^2}{b^4+c^4-3a^4-2b^2c^2+2a^2b^2+2a^2c^2}.$$

It is suggested that the reader use these ratio values in the equations of Theorem 2. By so doing the intercept ratios for a line through these two points may be calculated.

The reader may further select points P and P' from among the centroid, orthocenter, circumcenter, Brocard points, Nagel point, Steiner point, Tarry point, etc. If ratio values are known for the two points chosen (and they are known by the author), then the ratios associated with line MNO may be determined.

When using Theorems 1 and 2, it should be remembered that the segments involved are to be treated as directed quantities. If D lies between B and C, ratio BD/DC is considered positive. If D lies on BC extended, then ratio BD/DC must be considered negative. Similar comments apply to the other ratios involved in the results given.

Before closing, it should be noted that Theorem 1 has been previously recorded by the author [1]. Its derivation has been reproduced here so that the reader may have the complete argument leading to the statement of Theorem 2.

Reference

1. D. M. Bailey, A triangle theorem, School Science and Mathematics, 60 (1960) 281–4.

A DISSECTION PROBLEM

JOHN THOMAS, New Mexico State University

Three years ago Fred Richman asked me this intriguing question: "Can a rectangle be dissected into N nonoverlapping triangles, all having the same area, if N is an odd integer?" (Such dissections obviously exist for every even integer.) Richman had shown that the decomposition is impossible when $N=3$ or 5; that if the decomposition is possible for N it is possible for $N+2$; and that quadrilaterals with all angles arbitrarily close to 90° could be so dissected. (If we dissect the unit square by parallel horizontal lines into m rectangles of area $1/m$, bisect each rectangle with a diagonal, and discard the top triangle, we will have a quadrilateral which is the union of $2m-1$ congruent triangles and whose angles are, for large m, all very close to 90°.)

Since that time the problem has been discussed informally with a number of mathematicians and has appeared in the *American Mathematical Monthly* as Problem 5479, but thus far no one has been able to produce such a dissection or show that one cannot exist. Since Richman's problem seems to be quite difficult, and since it has not been discussed in the literature, perhaps the following theorems will be of interest.

Let $R(a, b)$ denote the rectangle with vertices (0, 0), $(a, 0)$, $(0, b)$, and (a, b). The transformation $x'=\lambda x$ $y'=\mu y$ is a dilatation of the plane which maps $R(a, b)$ onto $R(\lambda a, \mu b)$, multiplying all areas by the constant factor $\lambda\mu$, whence one concludes that the problem is independent of the shape and dimensions of the rectangle. Suppose that we have a decomposition of the unit square $R(1, 1)$ into N nonoverlapping triangles of area $1/N$. We can attach to it a 1 by $2/N$ rectangle which has been dissected by a diagonal into two triangles of area $1/N$, to get a dissection of $R(1+(2/N), 1)$ into $N+2$ nonoverlapping triangles of equal area. Thus we have:

THEOREM 1. *If Richman's problem has an affirmative solution for any odd integer N, it has an affirmative solution for every larger odd integer.*

If Richman's problem has an affirmative solution for the integer $N=nm$ with n and m odd integers, then $R(m, n)$ is decomposable into nonoverlapping triangles of unit area. Since a triangle of integral area k can obviously be dissected into k triangles of unit area, we see that Richman's problem is equivalent to this one: "Can $R(m, n)$ be dissected into nonoverlapping triangles of integral area, for odd integers m and n?"

corded by the author [1]. Its derivation has been reproduced here so that the reader may have the complete argument leading to the statement of Theorem 2.

Reference

1. D. M. Bailey, A triangle theorem, School Science and Mathematics, 60 (1960) 281–4.

A DISSECTION PROBLEM

JOHN THOMAS, New Mexico State University

Three years ago Fred Richman asked me this intriguing question: "Can a rectangle be dissected into N nonoverlapping triangles, all having the same area, if N is an odd integer?" (Such dissections obviously exist for every even integer.) Richman had shown that the decomposition is impossible when $N=3$ or 5; that if the decomposition is possible for N it is possible for $N+2$; and that quadrilaterals with all angles arbitrarily close to 90° could be so dissected. (If we dissect the unit square by parallel horizontal lines into m rectangles of area $1/m$, bisect each rectangle with a diagonal, and discard the top triangle, we will have a quadrilateral which is the union of $2m-1$ congruent triangles and whose angles are, for large m, all very close to 90°.)

Since that time the problem has been discussed informally with a number of mathematicians and has appeared in the *American Mathematical Monthly* as Problem 5479, but thus far no one has been able to produce such a dissection or show that one cannot exist. Since Richman's problem seems to be quite difficult, and since it has not been discussed in the literature, perhaps the following theorems will be of interest.

Let $R(a, b)$ denote the rectangle with vertices (0, 0), $(a, 0)$, $(0, b)$, and (a, b). The transformation $x'=\lambda x$ $y'=\mu y$ is a dilatation of the plane which maps $R(a, b)$ onto $R(\lambda a, \mu b)$, multiplying all areas by the constant factor $\lambda\mu$, whence one concludes that the problem is independent of the shape and dimensions of the rectangle. Suppose that we have a decomposition of the unit square $R(1, 1)$ into N nonoverlapping triangles of area $1/N$. We can attach to it a 1 by $2/N$ rectangle which has been dissected by a diagonal into two triangles of area $1/N$, to get a dissection of $R(1+(2/N), 1)$ into $N+2$ nonoverlapping triangles of equal area. Thus we have:

THEOREM 1. *If Richman's problem has an affirmative solution for any odd integer N, it has an affirmative solution for every larger odd integer.*

If Richman's problem has an affirmative solution for the integer $N=nm$ with n and m odd integers, then $R(m, n)$ is decomposable into nonoverlapping triangles of unit area. Since a triangle of integral area k can obviously be dissected into k triangles of unit area, we see that Richman's problem is equivalent to this one: "Can $R(m, n)$ be dissected into nonoverlapping triangles of integral area, for odd integers m and n?"

Let us designate the points of the plane whose coordinates are both integers as *lattice points*. A dissection will be called *simplicial* if the intersection of two triangles is a vertex or an edge of each triangle.

THEOREM 2. *If m and n are odd integers, there is no dissection of $R(m, n)$ into nonoverlapping triangles whose vertices are lattice points and whose areas are integral.*

Proof. We distinguish four types of lattice points; $0=$(even, even), $1=$(even, odd), $2=$(odd, even), and $3=$(odd, odd); and therefore ten types of line segments whose endpoints are lattice points, corresponding to the unordered pairs $\{i, j\}$, $0 \leqq i, j \leqq 3$. With each line segment, having vertices $v_1=(x_1, y_1)$ and $v_2=(x_2, y_2)$, we will associate the integer $\Delta(v_1, v_2)$ which is the value of the determinant $\left|\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right|$, and we will call the segment even or odd, depending on whether $\Delta(v_1, v_2)$ is even or odd. Since the area of the triangle having vertices v_1, v_2, and v_3 is equal to $\frac{1}{2}(\Delta(v_1, v_2)+\Delta(v_2, v_3)+\Delta(v_3, v_1))$, we see that a lattice triangle has nonintegral area if and only if it has exactly one or three odd sides. A straightforward calculation reveals that of the ten types of segments, only the types $\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$ are odd, the remainder are even.

There are twenty types of lattice triangles if we classify them by vertex type. Four of these types, $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, and $\{1, 2, 3\}$, have vertices of three different types. They will be called *complete*, and the remaining sixteen types will be called *incomplete*. Note that an incomplete triangle of type $\{a, a, b\}$ must have integral area since the $\{a, a\}$ side is even and the two $\{a, b\}$ sides have the same parity, while a complete triangle is easily checked to have one or three odd sides, hence nonintegral area. This establishes

LEMMA 1. *A lattice triangle has integral area if and only if it is incomplete.*

Every pair of lattice points determines a straight line whose equation is $Ax+By=C$ with A, B, and C integers having no common factor. Note that at least one of A and B must be odd, for if A and B were both even, $Ax+By$ would be even for any integers x and y, and therefore C would also have to be even. There are six possibilities for the equation $\overline{Ax}+\overline{By}=\overline{C}$ (where the bar of a number denotes its residue modulo two) and they are $\bar{x}=0$, $\bar{y}=0$, $\bar{x}=1$, $\bar{y}=1$, $\bar{x}+\bar{y}=0$, and $\bar{x}+\bar{y}=1$. Each of these six equations is satisfied by precisely two types of lattice points, as follows: $\bar{x}=0$ by types 0 and 1, $\bar{y}=0$ by types 0 and 2, $\bar{x}+\bar{y}=0$ by types 0 and 3, $\bar{x}+\bar{y}=1$ by types 1 and 2, $\bar{y}=1$ by types 0 and 3, and $\bar{x}=1$ by types 2 and 3. This establishes

LEMMA 2. *Any lattice point lying on a segment of type $\{a, b\}$ with $a \neq b$, is either of type a or type b.*

Now let $R(m, n)$ be dissected simplicially into lattice triangles. Since m and n are odd, the vertices and edges of $R(m, n)$ are all of different types—the vertical edges are of types $\{1, 1\}$ and $\{2, 3\}$ while the horizontal edges are of type $\{0, 2\}$ and $\{1, 3\}$. Thus Lemma 2 implies that in any decomposition of $R(m, n)$ into lattice triangles, the segments of type $\{0, 1\}$ must appear either inside $R(m, n)$ or on the $\{0, 1\}$ edge. It follows easily (see [1], p. 12) that there are an odd number of type $\{0, 1\}$ segments on the boundary of $R(m, n)$. Let us now consider each triangle of the decomposition in turn and place a mark on each

$\{0, 1\}$ edge of each triangle. There will be, all told, an odd number of marks, since each internal $\{0, 1\}$ edge abuts two triangles, hence receives two marks. On the other hand, each incomplete triangle contributes an even number of marks, so there must be at least one complete triangle of the type $\{0, 1, 2\}$ or $\{0, 1, 3\}$. This fact, together with Lemma 1, establishes Theorem 2 for simplicial decompositions.

We will now consider nonsimplicial dissections of $R(m, n)$, all of whose vertices are lattice points. We will say that a vertex of the dissection is a *bad vertex* if it fails to be a vertex of some triangle t on whose boundary it lies, and we will say in this case that it "lies on one edge" of t.

If Theorem 2 is false there is a lattice point dissection of $R(m, n)$, for certain odd integers m and n, which has no complete triangle, and has a minimum number of bad vertices. That is, our minimal dissection D has N bad vertices, and every lattice point dissection with fewer than N bad vertices has a complete triangle. N is at least 1, since a dissection without bad vertices is simplicial.

There are only four possible cases:

Case 1. Some bad vertex lies on one edge of a triangle of type $\{b, b, b\}$

Case 2. Some bad vertex lies on one edge of type $\{a, b\}$ of a triangle of type $\{a, b, b\}$. In this case, by Lemma 2, the bad vertex must have type a or type b.

Case 3. Some bad vertex of type a or b lies on one edge of type $\{b, b\}$ of a triangle of type $\{a, b, b\}$

Case 4. Every bad vertex is of type c and lies on one edge of type $\{b, b\}$ of a triangle of type $\{a, b, b\}$.

The letters a, b, and c stand for distinct vertex types. In cases 1, 2, or 3, we can dissect the triangle into two triangles by drawing a line from the bad vertex, to the vertex of the triangle opposite the side on which it lies. The dissection of $R(m, n)$ thus obtained will have $N-1$ bad vertices, and *no* complete triangle, since neither of the two new triangles can be complete. As this would contradict the minimality of the number of bad vertices, cases 1, 2 and 3 cannot occur.

In Case 4 we first perform the dilatation $\bar{x}=3x$, $\bar{y}=3y$ and consider the similar dissection of $R(3m, 3n)$; that is, $v=(x, y)$ was a vertex in the original dissection D of $R(m, n)$ if and only if $\bar{v}=(3x, 3y)$ is a vertex in the similar dissection $\overline{D}$ of $R(3m, 3n)$. The vertices v and $\bar{v}$ are of the same type, so $\overline{D}$ also has N bad vertices and no complete triangles. We consider a triangle of $\overline{D}$ with vertices P and Q of type b and R of type a and denote by S the bad vertex of type C, which lies on side PQ. Let T denote the point on side QR which is two thirds of the way from Q to R. If $Q=(3x, 3y)$, and $R=(3x_2, 3y_2)$, then $T=(x_1+2x_2, y_1+2y_2)$, whence T is a lattice point of type b. No bad vertex can lie on QR, so that QR is either a part of the boundary of $R(3m, 3n)$ or is an edge of a second triangle whose opposite vertex will be denoted by U. U is of type a or b. We draw the straight lines TS, TP, and (if necessary) TU. It is easy to check that this dissection will have $N-1$ bad vertices, and no complete triangles, whence Case 4 cannot occur either. This completes the proof of Theorem 2.

COROLLARY. *There is no dissection of the unit square into an odd number of nonoverlapping triangles having the same area, for which all of the vertices have rational numbers with odd denominators as coordinates.*

Proof. Let m be the least common multiple of the denominators of the abcissas of the vertices, n be the least common multiple of the ordinates of the vertices, and N the number of triangles. The transformation $x=nNx$, $y=mNy$ gives a decomposition of $R(mN, nN)$ into N lattice triangles of integral area mnN, which is impossible as Nn and Nm are odd.

We can improve this last result slightly as follows:

THEOREM 3. *Given odd integers m and n, let $S(m, n)$ be the subset of $R(1, 1)$ consisting of all points in squares of side $1/6nm$ centered at the points $v_{j,k}=((j/n), (k/m))$ and having sides parallel to the sides of $R(1, 1)$. There is no decomposition of $R(1, 1)$ into nonoverlapping triangles having the same area, for which all vertices lie in $S(m, n)$.*

Proof. Expanding, $S(m, n)$ becomes the subset $S^*(m, n)$ of $R(m, n)$ consisting of rectangles of width $1/6n$ and length $1/6m$ centered at lattice points. Suppose $v_1=(x_1, y_1)$ and $v_2=(x_2, y_2)$ lie in $S^*(m, n)$ and let $v_1^*=(x_1^*, y_1^*)$ and $v_2^*=(x_2^*, y_2^*)$ be the nearest lattice points to v_1 and v_2 respectively. Then $x_i=x_i^*+u_i$ and $y_i=y_i^*+w_i$ for $i=1$ and 2, where $|u_i|<(1/12n)$ and $|w_i|<(1/12m)$. Then

$$|\Delta(v_1, v_2) - \Delta(\overset{*}{v_1}, \overset{*}{v_2})| < \left|\det\begin{matrix} u_1 u_2 \\ y_1 y_2 \end{matrix}\right| + \left|\det\begin{matrix} \overset{*}{x_1}, & \overset{*}{x_2} \\ w_1, & w_2 \end{matrix}\right|$$

$$< \frac{1}{12n}|y_2 - y_1| + \frac{1}{12m}|\overset{*}{x_2} - \overset{*}{x_1}| < \frac{1}{6}$$

since $|y_2-y_1|<n$ and $|x_2^*-x_1^*|<m$ simply because these points all lie in $R(m, n)$. It follows that if we have a triangle T whose vertices v_1, v_2, and v_3 all belong to $S^*(m, n)$ then the triangle T^* with vertices v_1^*, v_2^*, v_3^* (v_i^* is the lattice point nearest v_i) has area differing from the area T by less than $1/2$.

Suppose that we have a decomposition of $R(m, n)$ into triangles $\{T_i\}$ of integral area having all vertices in $S^*(m, n)$. Now if we distort the decomposition slightly by moving all the vertices to the nearest lattice point, we will have a dissection of $R(m, n)$ into triangles $\{T_i^*\}$ such that $|\text{area } T_i - \text{area } T_i^*| < 1/2$. But area T_i^* is an integer multiple of $1/2$, since T_i^* is a lattice triangle, and area T_i is an integer, so area $T_i=$ area T_i^*. This is impossible by Theorem 2.

These theorems are the residue of an approach to Richman's problem which envisioned proving (1) that no decomposition of the desired sort exists, whose vertices are rational points of $R(1, 1)$ and (2) that any decomposition could be distorted slightly to yield a decomposition with rational points as vertices.

I suspect that (1) is true, but I have been unable to take care of the rationals with even denominator. The estimates in Theorem 3 are not very sharp, but the set $S(n, m)$ which is eliminated from the unit square has area $(1/36n^2m^2)$ for each n, m. I do not know the area of the union of all the sets $S(n, m)$ (n, m odd), as they overlap rather badly.

Reference

1. S. K. Stein, Mathematics: The Man-Made Universe, Freeman, San Francisco, 1963.

ANSWERS

A435. The ratio is one, since each sum is one. This follows from the known simple summation

$$1 = \frac{1}{1-a_1} - \frac{a_1}{(1-a_1)(1-a_2)} + \frac{a_1 a_2}{(1-a_1)(1-a_2)(1-a_3)} - \cdots$$

A436. It is easy to show that $1/n = 1/(n+a) + 1/(n+b)$ if and only if $ab = n^2$.

A437. Let $a_1, a_2, \cdots, a_n$ designate successive sides of the polygon and let x designate the portion of a_1 from the first vertex to its point of tangency. Then determine each succeeding segment between a vertex and a point of tangency in terms of x. The last section of a_n will equal the first segment of a_1. Thus $x = a_n - a_{n-1} + a_{n-2} - \cdots + a_1 - x$ or $x = \frac{1}{2}(a_n - a_{n-1} + a_{n-2} - \cdots + a_1)$. Hence x is rational. Since this argument will hold for any segment, it follows that each segment is rational.

A438. $N = 3^n n! = 3^n \cdot 1 \cdot 2 \cdot 3 \cdots n = 3 \cdot 6 \cdot 9 \cdots (3n)$. The numbers are one less and one more than the exhibited odd factors of N are even, so $(3n)!$ has n even factors other than those in N. Hence, $(3n)!/2^n N = (3n)!/(6^n n!)$ is a positive integer if n is.

A439. There is no reason why the letters should be distinct. Hence there are really $n \times n \times n$ instances to consider. But there is nothing to prove if one or more of the three elements is the identity. Also there is nothing to prove if the three elements are alike. Hence the total number of instances for which testing is needed is $(n-1)^3 - (n-1) = n(n-1)(n-2)$.

(Quickies on pages 223–224)

ABSOLUTER GEOMETRY

RICHARD MENZEL, Haile Sellassie I University

1. Introduction. A general plane synthetic geometry which we will call "absoluter geometry" may be postulated for which all absolute geometries and all singly elliptic geometries (and perhaps some other geometries) will serve as models.

The incidence axioms employed are those of projective geometry. The notion of separation of four concurrent lines is used as a primitive idea. The axioms regarding this separation are duals of those employed by Borsuk and Szmielew [1, pages 365–6] in their treatment of projective geometry, except for slight further modifications in Axioms 8 and 9 (see Section 2). Nothing is assumed regarding the presence, absence, or number of parallels to a line L which pass through a point not on L. The continuity axiom is that of Dedekind, a plane separation axiom is employed, and the congruence axioms are generalizations of those of Hilbert [2, pages 12–15], some of them depending upon generalizations of the usual definitions of ray and of angle.

The notions of separation and betweenness of collinear points are defined in terms of the (primitive) notion of separation of four concurrent lines.

We prove that the class of all singly elliptic geometries is precisely the class of all absoluter geometries in which every two lines intersect. Also the class of all absolute geometries is precisely the class of all absoluter geometries which satisfy the axiom that "through every point not on a line L there passes at least one line which does not intersect L". While proving these statements we also show that the set of geometries without parallel lines which satisfies Axioms 1–10 of absoluter geometry is the set of all projective geometries, and that the set of geometries satisfying Axioms 1–11 and the parallel axiom of absolute geometry is the set of all descriptive geometries satisfying that parallel axiom.

2. The axioms of absoluter geometry. Let the symbol $AB//CD$ be read as "lines A and B separate lines C and D."

Axiom 1. For any line L there are (at least) three distinct points on L.

Axiom 2. For every two points there is a unique line passing through them.

Axiom 3. There are (at least) three noncollinear points.

Axiom 4. If $AB//CD$, then A, B, C and D are concurrent and distinct.

Axiom 5. If $AB//CD$, then $AB//DC$ and $CD//AB$.

Axiom 6. If A, B, C, D are four distinct concurrent lines, then exactly one of the relations $AB//CD$, $AC//BD$, $AD//BC$ holds.

Axiom 7. For any concurrent and distinct lines A, B, C, D, E if not $AB//CD$ and not $AB//CE$, then not $AB//DE$.

DEFINITION. *$A_1A_2 \cdots A_n \overset{L}{\underset{\wedge}{=}} B_1B_2 \cdots B_n$ means that $A_1, A_2, \cdots, A_n$ are concurrent at a point p, $B_1, B_2, \cdots, B_n$ are concurrent at a point q, and there is a line L which does not pass through p or q but such that A_i, B_i and L are concurrent for $i=1, 2, \cdots, n$.*

Axiom 8a. If $AB//CD$ and $ABCD \overset{L}{\underset{\wedge}{=}} A'B'C'D'$, then $A'B'//C'D'$.

Axiom 8b. If $ABC \overset{L}{\underset{\wedge}{=}} A'B'C'$ and if $AB//CD$, where D is a line which does not intersect L, then $A'B'//C'D'$, if D' is a line concurrent with A', B', C' which does not intersect L.

Let us designate the line passing through points $x_1, x_2, \cdots, x_n$ by $\langle x_1x_2 \cdots x_n \rangle$.

Axiom 9. If a, b, p are noncollinear points and C is any line through p other than $\langle ap\rangle$ or $\langle bp\rangle$, there is a point d collinear with a and b such that $\langle bp\rangle C//\langle ap\rangle\langle dp\rangle$.

DEFINITION I. *Suppose a, b, c, and d are points on line L. The symbol $ab//cd$ is read "a and b separate c and d" and means that there is a point p not on L such that $\langle ap\rangle\langle bp\rangle//\langle cp\rangle\langle dp\rangle$.* [3, page 174.]

Separation of 4 collinear points is "independent of p" due to Axiom 8a.

DEFINITIONS. *If A, B, C are three concurrent lines, we define the pencil segment AB/C to be the class of lines X which satisfy $AB//CX$. The pencil segment AB/C together with the lines A, B is called a pencil interval and is denoted $[AB/C]$. The notions of line segment ab/c and line interval $[ab/c]$ are duals of the notions of pencil segment and pencil interval. A point d lies between two points x, y belonging to interval $[ab/c]$ if $xy//cd$.*

DEFINITION. *Suppose a, b, c are points on line L. The line segment ab/c and the line interval $[ab/c]$ are said to be unbroken if there is a point p not on L such that all lines of the pencil segment $\langle ap\rangle\langle bp\rangle/\langle cp\rangle$ intersect L.*

It is not difficult to show that two points a and b on line L determine two unbroken line intervals $[ab/c]$ and $[ab/d]$ if all lines intersect L and only one unbroken line interval $[ab/c]$ if the parallel axiom of absolute geometry holds.

Axiom 10 (Dedekind). For every partition of all of the points of an unbroken line segment ab/c into two nonvacuous sets, such that no point of either set lies between two points of the other (belonging to $[ab/c]$), there exists a point of one set which lies between every other point of that set and every point of the other set.

Axiom 11 (plane separation). The set complement of any line L equals the union of two sets A and B having the properties that (a) if x, y belong to one of the sets A or B then some unbroken interval $[xy/z]$ is contained in that set and (b) if x is a point in A and y is a point in B then all unbroken intervals of the form $[xy/z]$ will intersect L. (A and B, if nonnull, are called sides of L.)

DEFINITION. *Two unbroken line intervals $[ab/c]$ and $[ad/e]$ are said to be related if one contains the other.*

It is easily shown that relatedness of unbroken line intervals is an equivalence relation on any set of unbroken line intervals of the form $[ax/y]$, where a is fixed and a and x are distinct.

DEFINITIONS. *A ray $ab/c\rangle$ is the equivalence class of related unbroken line intervals of which $[ab/c]$ is a member and every member of which contains a. Rays $ab/c\rangle$ and $ad/e\rangle$ are said to be opposite rays if they are not the same but a, b, d are collinear.*

Axiom 12. An equivalence relation $\sim$ (called congruence) exists on the set of all unbroken line intervals.

Axiom 13. Let $[ab/c]$ be any unbroken line interval and let $de/f\rangle$ be any ray. There is a unique element of $de/f\rangle$ which is equivalent to $[ab/c]$.

DEFINITION. *A pencil interval $[AB/C]$ is referred to as an angle. $[AB/C]$ may also be designated $[AB/C]_p$ if p is the intersection of A, B, C.*

Axioms 14 and 15 are duals of 12 and 13 except that the word "unbroken" is omitted. The notion of ray may be dualized into that of "angular ray."

Axiom 14. An equivalence relation $\sim$ (called congruence) exists on the set of all angles.

Axiom 15. Let $[AB/C]$ be any angle and let p be a point on the line A'. There are exactly two angles of the form $[A'X/Y]_p$ which are congruent to $[AB/C]$ and neither contains the other.

Axiom 16. If $[BA/G]\sim[B'A'/G']$ and $[BC/F]\sim[B'C'/F']$ and $[BA/G]$ and $[BC/F]$ are in opposite angular rays as are $[B'A'/G']$ and $[B'C'/F']$ and $A\neq C$, then $[AC/Z]\sim[A'C'/Z']$ where $AC//BZ$ and $A'C'//B'Z'$, and $[AC/B]\sim[A'C'/B']$. If, furthermore, the union of $[BA/G]$ and $[BC/F]$ is a proper subset of the line pencil lying on the intersection of A and B, then the union of $[B'A'/G']$ and $[B'C'/F']$ is a proper subset of the line pencil lying on the intersection of A' and B'.

DEFINITION. *If a, b, c are noncollinear the union of unbroken line intervals $[ab/x]$, $[ac/y]$ and $[bc/z]$ is called a three-side and is designated (abc, xyz). $[ab/x]$, $[ac/y]$ and $[bc/z]$ are called the sides of the three-side, a, b, c are called its vertices, and $[\langle ac\rangle\langle bc\rangle/\langle xc\rangle]$ is called "the angle opposite side $[ab/x]$" (etc.). A three-side (abc, xyz) is called a Paschian three-side, an even three-side, or a triangle (and is designated $\Delta(abc, xyz)$), if every line which does not pass through a vertex intersects the three-side in an even number of points.*

Axiom 17 (side-angle-side). Given $\Delta(abc, xyz)$ and $\Delta(a'b'c', x'y'z')$, if $[bc/z]\sim[b'c'/z']$ and $[ac/y]\sim[a'c'/y']$ and $[\langle ac\rangle\langle bc\rangle/\langle xc\rangle]\sim[\langle a'c'\rangle\langle b'c'\rangle/\langle x'c'\rangle]$, then $[ab/x]\sim[a'b'/x']$ and $[\langle ab\rangle\langle cb\rangle/\langle yb\rangle]\sim[\langle a'b'\rangle\langle c'b'\rangle/\langle y'b'\rangle]$.

DEFINITION. *An absoluter geometry is a collection S of objects (points) together with a collection of subsets of S (lines), these lines and points satisfying Axioms* 1–17 *regarding the relations of incidence, separation of concurrent lines, and congruence of unbroken line intervals and of angles.*

3. Projective and singly elliptic geometries.

DEFINITIONS. *A projective geometry satisfies Axioms* 1, 2, 3, *and* 10, *the duals of Axioms* 4, 5, 6, 7, *and* 8a, *and the axioms that "all lines intersect" and "given three collinear points a, b, c there is a point d satisfying $bc//ad$." A projective geometry which satisfies Axioms* 12–17 *is called a singly elliptic geometry (the term "unbroken" may be deleted where it occurs in these axioms).*

THEOREM. *The set of absoluter geometries in which all lines intersect is identical with the set of all singly elliptic geometries.*

LEMMA. *The set of all projective geometries is identical with the set of geometries which satisfy Axioms* 1–10 *of absoluter geometry and "all lines intersect."*

LEMMA. *All singly elliptic geometries are absoluter geometries, subject to the dual of Definition I.*

Note. The plane separation axiom is a theorem in any projective geometry, one of the sets A and B corresponding to a line L being null. In a projective geometry every three-side may be shown to be even or "odd" but some are odd.

4. Descriptive and absolute geometries.

We now discuss the relation between

absolute and absoluter geometries and between descriptive geometries and geometries satisfying Axioms 1–11.

DEFINITION. *A descriptive geometry is a set S of objects (called points) together with a set of subsets of S (lines) which satisfy the following axioms regarding the relations of incidence and of betweenness.* (a) *There are at least two points on each line.* (b) *Axioms 2 and 3 of absoluter geometry.* (c) *If b is between points a and c (symbolically a-b-c), then a, b, c are collinear and distinct.* (d) *If a-b-c then c-b-a.* (e) *If a, b and c are collinear and distinct, then at least one of the relations a-b-c, b-a-c or a-c-b holds.* (f) *If a-b-c, then not b-a-c.* (g) *If a and b are distinct points, there is a point d such that a-b-d.* (h) *If a-b-c and b-c-d, then a-b-d.* (i) *If a-b-d and b-c-d, then a-b-c.* (j) *Calling the set of points between a and b the segment (ab) and (ab) together with a and b the interval $[ab]$, we assume that for every division of a segment (ab) into two nonvacuous sets, such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.* (k) *(Due to Pasch). If a, b, and c are noncollinear points, all lines which do not pass through a, b, or c intersect the triangle $\triangle\, abc$ (i.e., the union of $[ab]$, $[ac]$ and $[bc]$) an even number of times.* [**1**, pages 21, 26, 61 and 151; **3**, page 160].

DEFINITION II. *Let a, b, and c be points on line L. We say "b is between a and c" (symbolically a-b-c) if there is a point p not on L and a line L' which passes through p and which does not intersect L such that $\langle ap\rangle\langle cp\rangle//\langle bp\rangle L'$.*

Betweenness of three collinear points is "independent of p and of L' (through p)" because of Axiom 8b. Pasch's axiom has an interesting interpretation in terms of Definition II.

LEMMA. *Every geometry which satisfies the first eleven axioms of absoluter geometry and the axiom that "through each point not on a line L there passes (at least) one line which does not intersect L" is a descriptive geometry which satisfies that parallel axiom.*

DEFINITION. *An absolute geometry satisfies the axioms of descriptive geometry (axioms a–k) as well as the following.*

(l) *Through each point p not on a line L there passes at least one line which does not intersect L.* (m) *There is an equivalence relation $\sim$ (called congruence) defined on the set of all intervals $[xy]$.* (n) *If $[ab]\sim[a'b']$, $[bc]\sim[b'c']$, a-b-c, and a'-b'-c', then $[ac]\sim[a'c']$.* (o) *Let the equivalence class of those intervals of the form $[ax]$ which contain $[ab]$ or are contained in $[ab]$ be called "the ray $ab\rangle$". If $[ab]$ and $[de]$ are any intervals there is a unique element of $de\rangle$ which is congruent to $[ab]$.* (p) *An angle is a nonordered pair of rays $(ab\rangle, ac\rangle)$ where a, b, and c are noncollinear. There is an equivalence (congruence) relation $\sim$ defined on the set of all angles.* (q) *Let an angle $(ab\rangle, ad\rangle)$ be given and let $xy\rangle$ be any ray. There are exactly two angles $(xy\rangle, xv_i\rangle)$ $i=1, 2$ which are congruent to $(ab\rangle, ad\rangle)$ and, furthermore, there is an element $[xv]$ of $xv_1\rangle$ and an element $[xv']$ of $xv_2\rangle$ such that $[vv']$ intersects $\langle xy\rangle$.* (r) *If $(ab\rangle, ad\rangle)\sim(xy\rangle, xu\rangle)$ and $[ab]\sim[xy]$ and $[ad]\sim[xu]$, then $[bd]\sim[yu]$ and $(yx\rangle, yu\rangle)\sim(ba\rangle, bd\rangle)$.*

LEMMA. *Every absoluter geometry in which the above parallel axiom (Axiom 1) holds is an absolute geometry.*

The proof of the lemma is trivial once it has been observed that because of the parallel axiom (Axiom 1) the only unbroken line interval of the form $[ab/c]$ is the interval $[ab]$. For if p is not on $\langle ab\rangle$ and L' passes through p and does not intersect $\langle ab\rangle$, then any line C which passes through p and which satisfies $L'C//\langle ap\rangle\langle bp\rangle$ must intersect $\langle ab\rangle$ or Axiom 8b can be shown to be violated. For purposes of the lemma we define $(ab\rangle,\ ac\rangle)\sim(a'b'\rangle,\ a'c'\rangle)$ if $[\langle ab\rangle\langle ac\rangle/L]\sim[\langle a'b'\rangle\langle a'c'\rangle/L']$ where L is a parallel to $\langle bc\rangle$ through a and L' is a parallel to $\langle b'c'\rangle$ through a'.

LEMMA. *Every descriptive geometry which satisfies the parallel Axiom* 1 *also satisfies Axioms* 1–11 *of absoluter geometry. Every absolute geometry is an absoluter geometry.*

Proof of lemma. In a descriptive geometry we may define separation of four collinear points a, b, c, and d (symbolically $ab//cd$) to mean that either (a) a-c-b and c-b-d, (b) a-d-b and d-b-c, (c) b-c-a and c-a-d, or (d) b-d-a and d-a-c. Separation of four concurrent lines A, B, C, and D (designated $AB//CD$), in turn, means that there is a line L which intersects A, B, C, and D at four points a, b, c, and d respectively and $ab//cd$. To justify our definition of $AB//CD$ we show that if A, B, C, D and E are five lines concurrent at p there is a line which intersects all of them and does not pass through p. To show this we choose a point a (other than p) on line A. Then we choose a' such that a-p-a'. Let b be any point (other than p) on line B. By Pasch's axiom, C, D, and E each intersect (ab) or $(a'b)$. If C, D, and E all intersect the same segment, then we are done. If not, then one of them intersects one of the segments at a point x and the others intersect the other segment. Use Pasch's axiom on three-side axa'. Either the line passing through a and x or the line through a' and x will intersect all the lines A, B, C, D, and E.

To prove Axiom 8a it is useful to show that if $abcd \overset{a}{\underset{\wedge}{=}} a'b'c'd'$ and $ab//cd$ then $a'b'//c'd'$. To prove Axiom 8b we prove that if a, b, and c are points on line L and a-c-b, then $\langle ap\rangle\langle bp\rangle//\langle cp\rangle L'$, where p is any point not on L and L' is any line through p which does not intersect L. Pick a point x satisfying b-p-x. Use Pasch's axiom successively on the three-sides with vertices axb, xlb, xap and $aa'b$ to show that a-l-x, l-a'-b, a-a'-p and a'-c'-b, where l is the intersection of L' and $\langle ax\rangle$, a' is the intersection of $\langle lb\rangle$ and $\langle ap\rangle$, and c' is the intersection of $\langle lb\rangle$ and $\langle cp\rangle$. From l-a'-b and a'-c'-b we deduce that c'-a'-l. From c'-a'-l and a'-c'-b, in turn, we deduce $a'b//c'l$ so that $\langle ap\rangle\langle bp\rangle//\langle cp\rangle L$ holds by definition.

For the purposes of this lemma we define $[AB/C]\sim[A'B'/C']$ if the angle (pair of rays) whose rays are contained in A and B (respectively) and in one side of C is congruent to the angle whose rays are contained in A' and B' (respectively) and in one side of C'.

THEOREM. *The set of geometries which satisfy Axioms* 1–11 *and the parallel axiom of absolute geometry is the set of all descriptive geometries which satisfy that parallel axiom. The set of absoluter geometries which satisfy the parallel axiom of absolute geometry is the set of all absolute geometries.*

5. Commentary. The question arises whether or not there are absoluter geometries which are neither absolute nor singly elliptic. For instance, perhaps there are absoluter geometries in which some but not all lines possess parallels, or in which a line L possesses parallel(s) but not through every point not lying on L.

It is well known that a model for Euclidean geometry may be produced by selecting one line of a projective geometry, calling its points ideal points, considering a Euclidean line to be a subset of a projective line (i.e., a projective line with the ideal point removed), and defining the notion of point b being between points a and c in terms of b separating a and c from the ideal point on line $\langle abc\rangle$. [See, for instance **1**, page 370.] Meserve [**4**, pages 272–4], among others, discusses a model for hyperbolic geometry which is a subset of a projective geometry, hyperbolic lines being subsets of projective lines. The problem of producing a model for a Euclidean or hyperbolic geometry which is a subset of a singly elliptic geometry, however, Euclidean or hyperbolic lines being subsets of elliptic lines and the congruence relations of the Euclidean or hyperbolic geometries being subsets of the congruence relation for elliptic geometry, would appear to be complicated by some of the resulting triangles, the sum of whose angles would be at once greater than and not greater than two right angles. In view of this consideration the concept of ideal point has not been employed in this paper.

References

1. K. Borsuk and W. Szmielew, Foundations of Geometry, rev. Eng. ed., North Holland Publishing Co., Amsterdam, 1960.

2. David Hilbert, The Foundations of Geometry, 2nd ed., Open Court Publishing Co., Chicago, 1921.

3. H. S. M. Coxeter, Non-Euclidean Geometry, Mathematical Expositions No. 2, University of Toronto Press, Toronto, 1942.

4. B. E. Meserve, Fundamental Concepts of Geometry, Addison-Wesley, London and Reading, 1955.

REMARKS ON ASYMPTOTES

HARRY GONSHOR, Rutgers University

Like many other topics, asymptotes are dealt with poorly in a basic course in calculus. However, unlike other topics, asymptotes are not considered important enough to be rigorously discussed in more advanced courses. Thus even mathematicians are not clear as to what the precise definition should be.

We shall deal only with some facets of the problem. For example, we shall not search for an intrinsic coordinate free definition. In fact, for most of the paper we shall consider the question as to whether the line $y=0$ is an asymptote, and consider positive x only.

The typical textbook claims that $y=0$ is an asymptote to $y=f(x)$ if $\lim_{x\to\infty} f(x)=0$, at least when testing for asymptotes of rational functions. Sometimes the definition is changed in midstream to also require that $\lim_{x\to\infty} f'(x)=0$. It then becomes natural to ask "why stop at the first derivative?"

5. Commentary. The question arises whether or not there are absoluter geometries which are neither absolute nor singly elliptic. For instance, perhaps there are absoluter geometries in which some but not all lines possess parallels, or in which a line L possesses parallel(s) but not through every point not lying on L.

It is well known that a model for Euclidean geometry may be produced by selecting one line of a projective geometry, calling its points ideal points, considering a Euclidean line to be a subset of a projective line (i.e., a projective line with the ideal point removed), and defining the notion of point b being between points a and c in terms of b separating a and c from the ideal point on line $\langle abc \rangle$. [See, for instance **1**, page 370.] Meserve [**4**, pages 272–4], among others, discusses a model for hyperbolic geometry which is a subset of a projective geometry, hyperbolic lines being subsets of projective lines. The problem of producing a model for a Euclidean or hyperbolic geometry which is a subset of a singly elliptic geometry, however, Euclidean or hyperbolic lines being subsets of elliptic lines and the congruence relations of the Euclidean or hyperbolic geometries being subsets of the congruence relation for elliptic geometry, would appear to be complicated by some of the resulting triangles, the sum of whose angles would be at once greater than and not greater than two right angles. In view of this consideration the concept of ideal point has not been employed in this paper.

References

1. K. Borsuk and W. Szmielew, Foundations of Geometry, rev. Eng. ed., North Holland Publishing Co., Amsterdam, 1960.

2. David Hilbert, The Foundations of Geometry, 2nd ed., Open Court Publishing Co., Chicago, 1921.

3. H. S. M. Coxeter, Non-Euclidean Geometry, Mathematical Expositions No. 2, University of Toronto Press, Toronto, 1942.

4. B. E. Meserve, Fundamental Concepts of Geometry, Addison-Wesley, London and Reading, 1955.

REMARKS ON ASYMPTOTES

HARRY GONSHOR, Rutgers University

Like many other topics, asymptotes are dealt with poorly in a basic course in calculus. However, unlike other topics, asymptotes are not considered important enough to be rigorously discussed in more advanced courses. Thus even mathematicians are not clear as to what the precise definition should be.

We shall deal only with some facets of the problem. For example, we shall not search for an intrinsic coordinate free definition. In fact, for most of the paper we shall consider the question as to whether the line $y=0$ is an asymptote, and consider positive x only.

The typical textbook claims that $y=0$ is an asymptote to $y=f(x)$ if $\lim_{x\to\infty} f(x)=0$, at least when testing for asymptotes of rational functions. Sometimes the definition is changed in midstream to also require that $\lim_{x\to\infty} f'(x)=0$. It then becomes natural to ask "why stop at the first derivative?"

Textbooks mention the possibility of curves crossing asymptotes and usually give an example. However, what about the curve: $y=(\sin x/x)$? There is a vague feeling that $y=0$ is not an asymptote. This suggests that the definition should include the requirement that for $|x|$ sufficiently large, $f(x)\neq 0$. Further, in order to rule out $y=(2+\sin x)/x$ this can even be strengthened to require that for $|x|$ sufficiently large, $|f(x)|$ decreases strictly to 0.

We shall show that for rational functions luck permits the luxury of being careless since the weakest possible condition already implies the strongest possible one. On the other hand for more general functions there are examples illustrating various possibilities.

1. Rational Functions. First, since a rational function has only finitely many zeros and since the derivative of a rational function is rational, it is clear that a monotonicity condition is necessarily satisfied for $|x|$ sufficiently large.

We now investigate the derivative. For the sake of completeness, since the proofs are short, we shall consider vertical and oblique asymptotes as well as horizontal asymptotes in this section. We begin with a vertical asymptote $x=a$. The rational function then has the form $s/((x-a)^n\cdot t)$ where $x-a$ is prime to s and t. The derivative is $(-nts+(x-a)[ts'-t's])/((x-a)^{n+1}t^2)$. Thus it has the form $u/((x-a)^{n+1}v)$ where $x-a$ is prime to u and v. Hence $\lim_{x\to a} f'(x)=0$.

Now let $y=ax+b$ be an asymptote. Then $f(x)=ax+b+(r/s)$ where the degree of r is less than the degree of s. Then $f'(x)=a+(u/v)$ where deg. $v-$deg. u $=1+$deg. $s-$deg. r. Hence $\lim_{x\to\infty} f'(x)=a$.

Note that the second case includes horizontal as well as oblique asymptotes, i.e., the computation is certainly valid for $a=0$. By induction, compatibility occurs for all higher derivatives except for vertical asymptotes where these are irrelevant anyway.

2. Counterexamples. Note that the function $f(x)=(\sin x/x)$ does satisfy $\lim_{x\to\infty} f^n(x)=0$ since $f^n(x)$ necessarily has the form $(r \sin x+s \cos x)/t$ where r, s, and t are rational and deg $t>\max(\deg r, \deg s)$

On the other hand $f(x)=(1/x) \sin x^2$ satisfies $\lim_{x\to\infty} f(x)=0$ but not $\lim_{x\to\infty} f'(x)=0$. $f(x)=(1/x)[2+\sin x^2]$ satisfies the additional condition that $f(x)$ is never zero.

In order to obtain a function $f(x)$ which, in addition, is strictly decreasing, we define $g(x)=(9/4)-x$ if $x\leqq 2$ and, for every integer n such that $n\geqq 2$, $g(x)=\max((1/(n+1)^2), 1+(1/(n^2))-n^2|n+(1/n^2)-x|)$ if $n\leqq x\leqq n+1$. Now let $f(x)=\int_x^\infty g(y)dy$. (Roughly speaking $g(x)$ consists of sharper and sharper teeth lying on lower and lower plateaus.)

3. Final remarks. If $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to\infty} f'(x)$ exists then it is clear by the mean value theorem that $\lim_{x\to\infty} f'(x)=0$. The possibility $\lim_{x\to\infty} f'(x)=\infty$ or $\lim_{x\to\infty} f'(x)=-\infty$ is also ruled out. Using this, it is possible to give an alternative proof that $\lim_{x\to\infty} f'(x)=0$ for rational functions.

Finally, I leave the definition of an asymptote to those who are more dogmatically minded. At any rate, we have shown that for rational functions all conceivable definitions are equivalent.

MULTINOMIAL COEFFICIENTS

MORTON ABRAMSON, York University, Toronto

Introduction. Let n and r be positive integers. Then

$$(1)\qquad (x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} (n: k_1, k_2, \cdots, k_r) \prod_{i=1}^{r} (x_i)^{k_i}$$

where

$$(2)\qquad (n: k_1, k_2, \cdots, k_r) = \frac{n!}{k_1! k_2! \cdots k_r!}$$

with

$$(3)\qquad k_1 + k_2 + \cdots + k_r = n$$

is the well-known multinomial theorem. (Cf. Riordan [**5**, p. 3]. For more references with regard to the history of the theorem see [**9**].) When $r=2$ we have the binomial theorem. The numbers $(n: k_1, k_2, \cdots, k_r)$, $k_i \geqq 0$, with condition (3) are called multinomial coefficients. In particular $(n: k_1, k_2) = \binom{n}{k_1}$. The purpose of this note is to obtain in an elementary manner, without using the multinomial theorem, some formulae and relations concerning multinomial coefficients by using a combinatorial interpretation of the multinomial coefficients. In particular some of the well-known formulae involving binomial coefficients are given natural combinatorial generalizations. Many of the results in Section 2 and the relations at the end of Section 4 are believed new. For other formulae involving multinomial coefficients see [**2**], [**9**], and [**10**].

1. Riordan [**5**, p. 92] using the multinomial theorem notes the lemma: *The multinomial coefficient $(n: k_1, k_2, \cdots, k_r)$ is the number of ways of distributing n unlike objects into r unlike cells labelled $C_1, C_2, \cdots, C_r$ with cell C_i containing exactly k_i objects, $i=1, \cdots, r$.*

Proof. The number of ways of arranging along a straight line n balls, k_i of color K_i, $\sum_{i=1}^{r} k_i = n$, $k_i \geqq 0$, with color $K_i \neq$ color K_j, $i \neq j$, is, by elementary theory of permutations, $(n: k_1, k_2, \cdots, k_r)$. Arrange the n distinct objects along a straight line labelling them from left to right $1, 2, \cdots, n$. Consider now a particular arrangement of the n balls. We identify object number j with the ball in the jth position from the left in this particular arrangement, $j=1, \cdots, n$. Place into cell C_i all objects identified with balls of color K_i, $i=1, \cdots, r$. Each arrangement of the balls gives rise to a distinct distribution of the objects into the cells and hence the lemma follows.

The above proof, in fact, exhibits a one-to-one correspondence between n objects, k_i of a certain kind, $i=1, \cdots, r$ and the distributions of n unlike objects into r unlike cells, k_i into cell K_i, $i=1, \cdots, r$. The binomial coefficients are obtained when $r=2$. Of course, if we had assumed knowledge of the binomial coefficients at the outset the above lemma also follows easily.

We now prove two basic formulae. The first is

(4) $(n+1:k_1, k_2, \cdots, k_r)$

$$= \sum_{i=1}^{r} (n:k_1, k_2, \cdots, k_{i-1}, k_i - 1, k_{i+1}, \cdots, k_r),\ k_i > 0.$$

In the number of distributions $(n+1: k_1, k_2, \cdots, k_r)$ a particular object is either in cell C_i or in one of the remaining $r-1$ cells. The number of distributions with the particular object in cell C_i is $(n: k_1, k_2, \cdots, k_{i-1}, k_i-1, k_{i+1}, \cdots, k_r)$. Since the particular object has to be in one of the r cells, (4) follows.

The second formula is

(5) $(p+q:k_1, k_2, \cdots, k_r)$

$$= \sum_{\substack{j_1+\cdots+j_r=p \\ k_1+\cdots+k_r=p+q,\ p>0,\ q>0}} (p:j_1, j_2, \cdots, j_r)(q:k_1-j_1, k_2-j_2, \cdots, k_r-j_r).$$

To obtain (5) we may proceed as follows. The $p+q$ objects to be distributed may be represented by $\{a_1, \cdots, a_p, b_1, \cdots, b_q\}$. Cell C_i contains k_i objects consisting of j_i objects from $\{a_1, \cdots, a_p\}$ and k_i-j_i objects from $b_1, \cdots, b_q\}$ and there are $(p: j_1, \cdots, j_r)(q: k_1-j_1, \cdots, k_r-j_r)$ such distributions with j_i fixed, $i=1, \cdots, r$. Hence (5).

Formulas (4) and (5) are obtained in [**9**] by means of the multinomial theorem with (5) also given in [2]. Putting $r=2$ in (4) we have the simple relation $\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$ and putting $r=2$, (5) is Vandermonde's Theorem ([**5**, p. 9]),

$$\sum_{j=0}^{k}\binom{p}{j}\binom{q}{k-j}=\binom{p+q}{k}.$$

2. By an even (odd) combination from n is meant a combination consisting of an even (odd) number of elements from $\{1, \cdots, n\}$. The combination consisting of zero elements from n is considered an even combination. Denote by $E(n)$ the number of even combinations from n and by $0(n)$ the number of odd combinations from n. Now $E(n)=E(n-1)+0(n-1)$ since $E(n-1)$ counts the even combinations excluding the element n and $0(n-1)$ counts the even combinations including the element n. A similar argument gives $0(n)=0(n-1)+E(n-1)$. Hence

(6) $$E(n) = 0(n) = 2^{n-1}$$

with $E(n)=E\sum_{k=0}^{n/2}\binom{n}{2k}$ and $0(n)=\sum_{k=0}^{n/2}\binom{n}{2k+1}$. (See [**5**], p. 9 and [**7**], p. 44.)

Denote by $E_{r,j}(n)$ the number of ways of distributing n unlike objects into r unlike cells, $C_1, C_2, \cdots, C_r$, such that a certain set of j specified cells contain in total an even number of objects, with zero of the objects taken as an even number. The symbol $0_{r,j}(n)$ is the number of ways of doing this but with the total number of objects in the specified j cells being odd. Then

(7) $$E_{r,j}(n) - 0_{r,j}(n) = (r-2j)[E_{r,j}(n-1) - 0_{r,j}(n-1)].$$

We may obtain (7) as follows. The number of distributions counted in $E_{r,j}(n)$

but with a particular object somewhere in the specified j cells is $(j)0_{r,j}(n-1)$. Those counted in $E_{r,j}(n)$ with the particular object somewhere in the remaining $r-j$ cells is $(r-j)E_{r,j}(n-1)$. Hence,

$$E_{r,j}(n) = (j)0_j(n-1) + (r-j)E_{r,j}(n-1).$$

Similarly it may be seen that

$$0_{r,j}(n) = (j)E_{r,j}(n-1) + (r-j)0_{r,j}(n-1).$$

By subtracting the last equation from the former, (7) is obtained.

Some special cases of (7) are

$$E_{2p,p}(n) - 0_{2p,p}(n) = 0, \tag{8}$$

$$E_{2p+1,p}(n) - 0_{2p+1,p}(n) = E_{2p+1,p}(n-1) - 0_{2p+1,p}(n-1),\ n > 1, \tag{9}$$

$$E_{2p+1,p+1}(n) - 0_{2p+1,p+1}(n) = -\left[E_{2p+1,p+1}(n-1) - 0_{2p+1,p+1}(n-1)\right],\ n > 1. \tag{10}$$

Since (9) and (10) are identities in n for a fixed p and since

$$E_{2p+1,p}(1) = p+1,\ 0_{2p+1,p}(1) = p,\ E_{2p+1,p+1}(1) = p,\ 0_{2p+1,p+1}(1) = p+1,$$

we obtain from (9) and (10)

$$E_{2p+1,p}(n) - 0_{2p+1,p}(n) = 1, \tag{11}$$

and

$$E_{2p+1,p+1}(n) - 0_{2p+1,p+1}(n) = (-1)^n. \tag{12}$$

Formulas (8) and (11) are given as the third summation formula of **[9]**,

$$\sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} (-1)^{(k_2+k_4+\cdots+k_{2p})}(n:k_1,\cdots,k_r) = \begin{cases} 0 \text{ if } r \text{ is even} \\ 1 \text{ if } r \text{ is odd,} \end{cases}$$

with $\sum_{i=1}^{r} k_i = n$, $k_i \geqq 0$ and where p is the largest integer such that $2p \leqq n$.

Denote by $N(n, r)$ the number of ways of distributing n unlike objects into r unlike cells. Then it is well known from elementary theory of permutations that,

$$N(n, r) = r^n. \tag{13}$$

Hence $E_{r,j}(n) + 0_{r,j}(n) = r^n$ and from (8) we have

$$E_{2p,p}(n) = 0_{2p,p}(n) = \frac{(2p)^n}{2}. \tag{14}$$

Putting $p=1$ in (14) we obtain (6). Also, using (11) and (12), we have

$$E_{2p+1,p}(n) = \frac{1 + (2p+1)^n}{2} \tag{15}$$

and

$$E_{2p+1,p+1}(n) = \frac{(-1)^n + (2p+1)^n}{2}. \tag{16}$$

The left hand sides of (15) and (16), of course, may be expressed as sums over multinomial coefficients.

Denote by $N_{m,w}(n, r)$ the number of ways of distributing n unlike objects into r unlike cells with a specified set of w of the cells always containing exactly m of the objects. Then

$$N_{m,w}(n, r) = \binom{n}{m} w^m (r - w)^{n-m} \tag{17}$$

since m objects may be chosen in $\binom{n}{m}$ ways, placed in w cells in $N(m, w)$ ways, and the remaining $n-m$ objects placed in the remaining $r-w$ cells in $N(n-m, r-w)$ ways. Using (13), (17) follows, with $N_{m,w}(m, w)=1$. Hence

$$\sum_{\substack{k_1+\cdots+k_r=n \\ k_1+\cdots+k_w=m}} (n: k_1, \cdots, k_w, k_{w+1}, \cdots, k_r) = \binom{n}{m} w^m (r - w)^{n-m}, \qquad r > w. \tag{18}$$

Putting $w=1$, $r=3$ in (18) we obtain

$$\begin{aligned} \sum_{k=0}^{n-m} (n: m, k, n - k - m) &= \sum_{k=0}^{n-m} \binom{n}{k+m}\binom{k+m}{m} \\ &= \sum_{s=m}^{n} \binom{n}{s}\binom{s}{m} = \binom{n}{m} 2^{n-m}. \end{aligned} \tag{19}$$

Denote by $R_{m,j}(n, r)$ the number of distributions of n unlike objects into r unlike cells, such that the first cell always contains exactly m objects and a specified set of j of the remaining $r-1$ cells always contains in total an even number of objects. $S_{m,j}(n, r)$ denotes those distributions having an odd number in the same specified set of j cells with m objects always in the first cell. If $m=n$, $R_{m,j}(m, r)=1$ and $S_{m,j}(m, r)=0$. Clearly for $m<n$

$$R_{m,j}(n,r) = \binom{n}{m} E_{r-1,j}(n - m) \quad \text{and} \quad S_{m,j}(n,r) = \binom{n}{m} 0_{r-1,j}(n - m), \tag{20}$$

the numbers $E_{r,j}(n)$, $0_{r,j}(n)$ discussed previously. Putting $r=2p+1$, $j=p$ in (20) and using (8) it follows that

$$R_{m,p}(n, 2p + 1) - S_{m,p}(n, 2p + 1) = \begin{cases} 0 & \text{if } m < n \\ 1 & \text{if } m = n. \end{cases} \tag{21}$$

Using (17) with $w=1$, and (21), we obtain

$$R_{m,p}(n, 2p + 1) = S_{m,p}(n, 2p + 1) = \frac{\binom{n}{m}(2p)^{n-m}}{2}, \qquad m < n. \tag{22}$$

Using multinomial coefficients and (21), we obtain

$$\sum_{k_2+k_3+\cdots+k_{2p+1}=n-m} (-1)^{k_2+k_4+\cdots+k_{2p}} (n: m, k_2, k_3, \cdots, k_{2p+1}) = \begin{cases} 0 & \text{if } m < n \\ 1 & \text{if } m = n. \end{cases} \tag{23}$$

If we had the condition that the first w cells must always contain exactly m objects instead of just the first, then corresponding results to (20), (21), and (23) readily follow. Putting $p=1$ in (23) we obtain, with change of variables as in (19),

$$\begin{aligned}\sum_{k=0}^{n-m}(-1)^k(n: m, k, n-k-m) &= \sum_{s=m}^{n}(-1)^{s-m}\binom{n}{s}\binom{s}{m} \\ &= \begin{cases}0 & \text{if } m\neq n \\ 1 & \text{if } m=n,\end{cases}\end{aligned}\tag{24}$$

in agreement with [9, p. 1061]. Note that (19) and (24) "differ" only by the factor $(-1)^{s-m}$. In [8] the numbers A_n^m and B_n^m, n and m being positive integers or zero, are defined to be quasi-orthogonal if $\sum_{s=m}^{n} A_n^s B_s^m=\delta_n^m$, where δ_n^m is the Kronecker delta. Hence the numbers $A_s^n=\binom{n}{s}$ and $B_s^m=(-1)^{s-m}\binom{s}{m}$ are quasi-orthogonal.

3. Denote by $f(n, r)$ the number of distributions of n unlike objects into r unlike cells with no cell empty. Now, for a fixed choice of i of the r cells, there are $N(n, r-i)$ ways of distributing the n objects in the remaining $r-i$ cells without restrictions. Using the well-known principle of inclusion and exclusion [5, p. 51] we have

$$f(n, r) = \sum_{i=0}^{r-1}(-1)^i\binom{r}{i}N(n, r-i).$$

Using (13) we obtain

$$f(n, r) = \sum_{i=0}^{r-1}(-1)^i\binom{r}{i}(r-i)^n,\tag{25}$$

in agreement with [1] wherein $f(n, r)$ is obtained by solving the difference equation

$$f(n, r) = r[f(n-1, r-1)+f(n-1, r)], \qquad n, r \text{ positive integers},\tag{26}$$

with $f(n, r)=0$ if $n<r$ and $f(n, 1)=1$ for every n. We may obtain (26) by noting that $rf(n-1, r-1)$ is the number of distributions counted in $f(n, r)$ but with the first object always being alone in a cell, while $rf(n-1, r)$ counts those where the first object is never alone in a cell. Four other references to (25) and (26) are given by Church in [1].

In terms of multinomial coefficients we have

$$\begin{aligned}f(n, r) &= \sum_{k_1+\cdots+k_r=n, k_i>0}(n: k_1, \cdots, k_r) \\ &= \sum_{i=0}^{r-1}(-1)^i\binom{r}{i}(r-i)^n.\end{aligned}\tag{27}$$

Clearly $\sum_{j=1}^{r}\binom{r}{j}f(n, j)=N(n, r)=r^n$ (using 13) and hence we have the identity

$$\sum_{j=1}^{r}\sum_{i=0}^{j-1}(-1)^i\binom{r}{j}\binom{j}{i}(j-i)^n = r^n. \tag{28}$$

Of course, $S(n,r)=f(n,r)/r!$ is the number of distributions of n unlike objects into r like cells with no cell empty, $S(n,r)$ being the Stirling number of the second kind.

4. Finally using multinomial coefficients we consider two known results. Firstly define

$$((n\colon a_1, \cdots, a_w)) = \frac{n!}{(1!)^{a_1}(2!)^{a_2}\cdots (w!)^{a_w}} \tag{29}$$

with $\sum_{i=1}^{w} ia_i = n$. Then (29) is the multinomial coefficient giving the number of distributions of n unlike objects into $r=\sum_{i=1}^{w} a_i$ unlike cells with a fixed set of a_i of r cells containing i objects, $i=1, \cdots, w$.

Denote by $\langle n\colon a_1, \cdots, a_w\rangle$ the number of ways of distributing n unlike objects into $\sum_{i=1}^{w} a_i$ like cells with exactly a_i of the cells containing i objects each. Then it is easy to see that

$$\begin{aligned}\langle n\colon a_1, \cdots, a_w\rangle &= \frac{((n\colon a_1, \cdots, a_w))}{a_1!\cdots a_w!}\\ &= \frac{n!}{(1!)^{a_1}\cdots (w!)^{a_w}a_1!\cdots a_w!}\end{aligned} \tag{30}$$

in agreement with [3]. The numbers $P(n, w)=\sum\langle n\colon a_1, \cdots, a_w\rangle$, summation taken over solutions $(a_1, \cdots, a_w)$ satisfying $\sum_{i=1}^{w} ia_i=n$, giving the number of distributions of n unlike objects into an unrestricted number of like cells with no cell containing more than w objects are treated in [3] wherein many references are also given. In particular there is a large amount of literature dealing with the number $P(n, n)$. (See, for example, [6] which also contains an extensive reference list.)

Denote by $\langle\langle n\colon a_1, \cdots, a_w\rangle\rangle$ the number of permutations of $1, 2, \cdots, n$ each consisting of cycle structure $[1^{a_1}, 2^{a_2}, \cdots, w^{a_w}]$. That is, each permutation consists of $\sum_{i=1}^{w} a_i$ cycles, a_i of length i, $i=1, \cdots, w$ and necessarily $\sum_{i=1}^{w} ia_i=n$. Then

$$\langle\langle n\colon a_1, \cdots, a_w\rangle\rangle = \langle n\colon a_1, \cdots, a_w\rangle \prod_{i=1}^{w}[(i-1)!]^{a_i} \tag{31}$$

and using (29) and (30) we obtain

$$\langle\langle n\colon a_1, \cdots, a_w\rangle\rangle = \frac{n!}{1^{a_1}2^{a_2}\cdots w^{a_w}a_1!a_2!\cdots a_w!}. \tag{32}$$

(Cf. [4], p. 9 and [5], p. 67.) Relation (31) follows by noting that the i objects or i integers in the ith cell, involved in a distribution counted in $\langle n\colon a_1, \cdots, a_w\rangle$, gives rise to $(i-1)!$ permutations of cycle length i.

Corresponding to the relations involving multinomial coefficients are relations involving the numbers $\langle n: a_1, \cdots, a_w\rangle$ and $\langle\langle n: a_1, \cdots, a_w\rangle\rangle$. For example, for $n>1$,

$$(33)\quad ((n: a_1, \cdots, a_w)) = \sum_{i=1}^{w} a_i((n-1: a_1, a_2, \cdots, a_{i-1}+1, a_i-1, a_i, \cdots, a_w))$$

with the understanding that $((n: a_1, \cdots, a_w))=0$ if $a_i=-1$ for any $i=1, \cdots, w$, follows immediately from (4).

From (30) and (33) it follows that, for $n>1$,

$$(34)\quad \langle n: a_1, \cdots, a_w\rangle = \sum_{i=1}^{w} (a_{i-1}+1)\langle n-1: a_1, \cdots, a_{i-1}+1, a_i-1, \cdots, a_w\rangle$$

with a similar understanding as for (33). The first term of the right side of (34) is $\langle n-1: a_1-1, a_2, \cdots, a_w\rangle$.

From (31) and (34) it follows that, for $n>1$,

$$(35)\quad \begin{aligned}&\langle\langle n: a_1, \cdots, a_w\rangle\rangle\\ &\quad = \langle\langle n-1: a_1-1, a_2, \cdots, a_w\rangle\rangle\\ &\qquad + \sum_{i=2}^{w} (a_{i-1}+1)(i-1)\langle\langle n-1: a_1, \cdots, a_{i-1}+1, a_i-1, \cdots, a_w\rangle\rangle\end{aligned}$$

with a similar understanding as for (33).

References

1. M. T. L. Bizley and C. A. Church, Jr., Solution to problem E 1837, Amer. Math. Monthly, 74 (1967) 439.

2. Leonard Carlitz, Sums of products of multinomial coefficients, Elem. Math., 18 (1963) 37–39.

3. F. L. Miksa, L. Moser and M. Wyman, Restricted partitions of finite sets, Canad. Math. Bull., 1 (1958) 87–96.

4. F. D. Murnaghan, The Theory of Group Representations, Johns Hopkins Press, Baltimore, 1938.

5. John Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958.

6. Gian-Carlo Rota, The number of partitions of a set, Amer. Math. Monthly, 71 (1964) 498–504.

7. I. J. Schwatt, An Introduction to the Operations with Series, University of Pennsylvania Press, 1924.

8. S. Tauber, On quasi-orthogonal numbers, Amer. Math. Monthly, 69 (1962) 365–372.

9. ———, On multinomial coefficients, Amer. Math. Monthly, 70 (1963) 1058–1063.

10. ———, Summation formulae for multinomial coefficients, The Fibonacci Quarterly, 3 (1965) 96–100.

THE DIHEDRAL GROUP OF LINEAR TRANSFORMATIONS IN THE PLANE

G. H. LUNDBERG, Vanderbilt University

In this paper we develop a process for the formation of dihedral groups, and show, as a specific example, the development of the group of order thirty-six. We give also the general case of the development of the dihedral group of order n.

Formation of dihedral groups. If a sphere is tangent at the origin of a set of coordinate axes in a complex plane, each point on the sphere can be made to correspond to a single point on the plane. The uppermost point on the sphere is taken as the center of projection. A line determined by a point on the sphere and the center of projection will intersect the complex plane, establishing a one-to-one correspondence between points on the surface of the sphere and points on the plane. The one-to-one correspondence between point P on the plane and point P' on the sphere is illustrated in Figure 1.

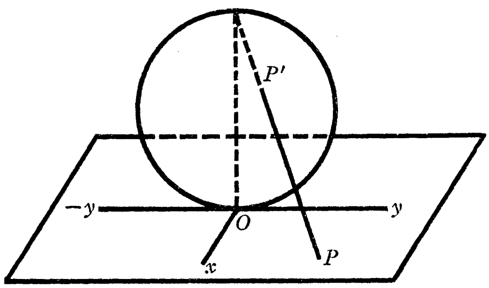


FIG. 1

If the sphere is rotated about an axis perpendicular to the plane, the invariant points in the plane will be $z=0$ and $z=\infty$.

In the linear fractional transformation

$$z' = \frac{az+b}{cz+d} \quad \text{where } ad-bc \neq 0,$$

the quadratic $cz^2+(d-a)z-b=0$ gives the invariant points. Since one invariant point is infinity, we have $c=0$; and since the other is zero, we have $b=0$ with $a \neq d$. The linear fractional transformation corresponding to any rotation of the sphere with the axis perpendicular to the complex plane at the origin is then

$$z' = \frac{a}{d} z,$$

which is the general linear transformation for any dihedral group.

The dihedral group of order thirty-six. As a specific example, a dihedral group of three rays will be considered. When the radius of the sphere is taken to be one-half unit in length, any projection of a pole of an axis of rotation parallel to the plane will lie at a distance of one unit from the origin on the complex plane. Figure 2 illustrates a circle with a radius 1 described about the origin as center, with three rays from the center intersecting the circumference at points 1, ω, and ω^2.

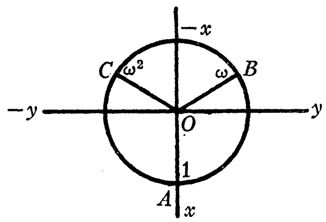


FIG. 2

The rotation of the sphere which carries A into B is found by substituting the value of z at A, which is 1, for z in $z' = (a/d)z$. Since the point A is transformed into B, we have $a = \omega d$. Substituting this value of a in $z' = (a/d)z$ gives $z' = \omega z$, which is the required transformation. It also carries B into C and C into A. Likewise, the rotation of the sphere which carries A into C and C into B is $z' = \omega^2 z$.

The rotation that carries A, B, and C into themselves is the identity, $z' = z$.

If the sphere is rotated about an axis parallel to the x-axis or the axis of reals in the complex plane, the fixed points in the plane will be 1 and -1. The linear fractional transformation corresponding to any rotation of the sphere with the axis in this position can be found by observing that we have $cz^2 + (d-a)z - b = 0$, whence $d = a$ and $c = b$.

The general linear fractional transformation for this rotation is then

$$z' = \frac{az + b}{bz + a} \cdot$$

If the rotation of the sphere is to carry B into C with the axis of rotation parallel to the axis of reals, the value ω for point B is substituted for z in the above linear fractional transformation. Then the result is equated to the value ω^2, at point C. Thus

$$\frac{a\omega + b}{b\omega + a} = \omega^2,$$

whence $a = 0$. Substituting this value of a in the general linear fractional transformation gives $z' = 1/z$, the transformation which carries B and C each into the other.

Any rotation of the sphere about an axis parallel to the line OB in the complex plane is

$$z' = \frac{az + c\omega^2}{cz + a},$$

since the values of z for the fixed points are ω and $-\omega$. Now, if point C and point A are carried each into the other, we have $z' = (\omega^2/z)$.

If a like method is used to carry points A and B each into the other when the axis of rotation is parallel to the line OC in the complex plane, the transformation becomes $z'' = \omega/z$. The entire set of transformations constitutes a group, as seen in Table 1.

TABLE 1. THE DIHEDRAL GROUP OF ORDER THIRTY-SIX

	$z' = z$	$z' = \omega z$	$z' = \omega^2 z$	$z' = \frac{1}{z}$	$z' = \frac{\omega}{z}$	$z' = \frac{\omega^2}{z}$
$z' = z$	$z' = z$	$z' = \omega z$	$z' = \omega^2 z$	$z' = \frac{1}{z}$	$z' = \frac{\omega}{z}$	$z' = \frac{\omega^2}{z}$
$z' = \omega z$	$z' = \omega z$	$z' = \omega^2 z$	$z' = z$	$z' = \frac{\omega^2}{z}$	$z' = \frac{1}{z}$	$z' = \frac{\omega}{z}$
$z' = \omega^2 z$	$z' = \omega^2 z$	$z' = z$	$z' = \omega z$	$z' = \frac{\omega}{z}$	$z' = \frac{\omega^2}{z}$	$z' = \frac{1}{z}$
$z' = \frac{1}{z}$	$z' = \frac{1}{z}$	$z' = \frac{\omega}{z}$	$z' = \frac{\omega^2}{z}$	$z' = z$	$z' = \omega z$	$z' = \omega^2 z$
$z' = \frac{\omega}{z}$	$z' = \frac{\omega}{z}$	$z' = \frac{\omega^2}{z}$	$z' = \frac{1}{z}$	$z' = \omega^2 z$	$z' = z$	$z' = \omega z$
$z' = \frac{\omega^2}{z}$	$z' = \frac{\omega^2}{z}$	$z' = \frac{1}{z}$	$z' = \frac{\omega}{z}$	$z' = \omega z$	$z' = \omega^2 z$	$z' = z$

The subgroups are

$$z' = \frac{1}{z}, \qquad z' = \frac{\omega}{z}, \qquad z' = \frac{\omega^2}{z}; \qquad z' = z, \qquad z' = \frac{1}{z};$$

$$z' = z, \qquad z' = \frac{\omega}{z}; \qquad \text{and} \qquad z' = z, \qquad z' = \frac{\omega^2}{z}.$$

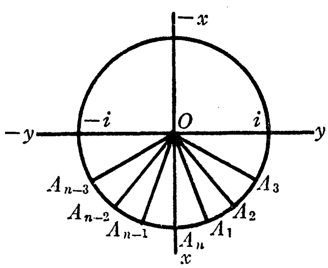


FIG. 3

The dihedral group of order n. With any dihedral group, the rotations of the sphere can be made to correspond to the fractional transformations in the complex plane. Figure 3 illustrates a dihedral group consisting of n distinct rays.

It was found that the rotation of the sphere about the perpendicular axis corresponds to the general linear fractional transformation $z'=(a/d)z$. If $R^1, R^2, \cdots, R^n$ represent the n roots of unity where

$$R^1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

$$R^2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n},$$

$$\cdots\cdots\cdots\cdots$$

$$R^n = \cos 2\pi + i\sin 2\pi = 1,$$

then $R^n=1$ represents a position in the complex plane on the x-axis at A_n, and R^1, R^2, etc., represent positions A_1, A_2, etc., in the complex plane, each at a distance of one unit from the origin. The transformation which carries A_n into A_1 or A_1 into A_2, etc., becomes $z'=Rz$. The rotations which carry A_n into A_2 and A_n into A_3 have the respective transformations $z'=R^2z$ and $z'=R^3z$. The transformation which carries A_n into itself is $z'=z$, which corresponds to one complete revolution of the sphere. All the transformations of this set will form a group.

Since the projections on the complex plane of the poles are always at a distance of one from the origin, any position along the circumference of the unit circle has a corresponding position at the other end of its diameter. Therefore the transformation

$$z' = \frac{az+b}{bz+a}$$

is general for any rotation of the sphere when the axis is parallel to the plane because the roots of the quadratic $cz^2+(d-a)z-b=0$ will differ only in sign.

A rotation of 180° sends A_{n-1} into A_1, and A_{n-2} into A_2, etc. If the axis of the sphere is parallel to the x-axis, then

$$\frac{aR^{n-1}+b}{bR^{n-1}+a} = R^1,$$

whence $a=0$. Hence the transformation is $z'=1/z$. Another rotation of 180° would bring the points back to their original positions. So, considered with the identity transformation $z'=z$, a subgroup of order two is formed.

The rotation of the sphere which carries A_{n-2} into A_n or A_{n-3} into A_1, etc., has an axis of rotation parallel to the line OA_{n-1}. The projections of poles give as values for z

$$\cos\frac{(n-1)2\pi}{n} + i\sin\frac{(n-1)2\pi}{n} = R^{n-1}$$

and

$$-\cos\frac{(n-1)2\pi}{n} - i\sin\frac{(n-1)2\pi}{n} = -R^{n-1}$$

as invariant points. Then the quadratic $cz^2+(d-a)z-b=0$ becomes $cz^2-b=0$. Since the roots differ only in sign, substituting either value of z for the fixed points gives $cR^{2n-2}-b=0$, whence $b=cR^{2n-2}$. The transformation corresponding to any rotation about an axis parallel to the line OA_{n-1} is

$$z' = \frac{az + cR^{2n-2}}{cz + a}.$$

If A_{n-2} is carried into A_n,

$$\frac{aR^{n-2} + cR^{2n-2}}{cR^{n-2} + a} = R^n,$$

whence $a=0$. Then if A_{n-2} is carried into $A_n, z'=(R^{n-2}/z)$, another rotation of 180° about the same axis would put the points in their original positions. So with the identity transformation $z'=z$, a subgroup of order two is formed.

In like manner, it can be shown that if the axis of rotation is placed parallel to each of the other rays in the complex plane, the transformation corresponding to each position will be included in the set of transformations,

$$z' = \frac{1}{z}, \quad z' = \frac{R}{z}, \quad z' = \frac{R^2}{z}, \quad \cdots, \quad z' = \frac{R^{n-1}}{z},$$

since there are n roots of unity. With the identity element each will form a subgroup of order two. Finally, the entire set of transformations will form a group.

References

1. Harold Coxeter, The Real Projective Plane, McGraw-Hill, New York, 1949.
2. Karl Doehlemann, Geometrische Transformationen, W. De Gruyter, Leipzig, 1930.
3. Boyd Patterson, Projective Geometry, Wiley, New York, 1937.
4. Oswald Veblen and John Young, Projective Geometry, Ginn, New York, 1918.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics San Jose State College, San Jose, California 95114.

BRIEF MENTION

Information. Freeman, San Francisco, 1966. vii+218 pp. $5.00 (cloth), $2.50 (paper).

The September 1966 issue of *Scientific American* was devoted entirely to this topic; included are all of the articles and many of the illustrations from that issue. A fascinating exposition of hardware, software, time-sharing, applications, artificial intelligence, etc., by such outstanding contributors as McCarthy, Greenberger, Minsky, Oettinger and Suppes.

Digital Computer Programming: Logic and Language. By C. M. Thatcher and A. J. Capato. Addison-Wesley, Reading, Mass., 1967. xi+159 pp. $3.95 (paper).

"Designed for a freshman-level programming course. . . . The approach is based on the conviction that programming *logic* is more important than programming language, as invariant points. Then the quadratic $cz^2+(d-a)z-b=0$ becomes $cz^2-b=0$. Since the roots differ only in sign, substituting either value of z for the fixed points gives $cR^{2n-2}-b=0$, whence $b=cR^{2n-2}$. The transformation corresponding to any rotation about an axis parallel to the line OA_{n-1} is

$$z' = \frac{az + cR^{2n-2}}{cz + a}.$$

If A_{n-2} is carried into A_n,

$$\frac{aR^{n-2} + cR^{2n-2}}{cR^{n-2} + a} = R^n,$$

whence $a=0$. Then if A_{n-2} is carried into $A_n, z'=(R^{n-2}/z)$, another rotation of 180° about the same axis would put the points in their original positions. So with the identity transformation $z'=z$, a subgroup of order two is formed.

In like manner, it can be shown that if the axis of rotation is placed parallel to each of the other rays in the complex plane, the transformation corresponding to each position will be included in the set of transformations,

$$z' = \frac{1}{z}, \quad z' = \frac{R}{z}, \quad z' = \frac{R^2}{z}, \quad \cdots, \quad z' = \frac{R^{n-1}}{z},$$

since there are n roots of unity. With the identity element each will form a subgroup of order two. Finally, the entire set of transformations will form a group.

References

1. Harold Coxeter, The Real Projective Plane, McGraw-Hill, New York, 1949.
2. Karl Doehlemann, Geometrische Transformationen, W. De Gruyter, Leipzig, 1930.
3. Boyd Patterson, Projective Geometry, Wiley, New York, 1937.
4. Oswald Veblen and John Young, Projective Geometry, Ginn, New York, 1918.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics San Jose State College, San Jose, California 95114.

BRIEF MENTION

Information. Freeman, San Francisco, 1966. vii+218 pp. $5.00 (cloth), $2.50 (paper).

The September 1966 issue of *Scientific American* was devoted entirely to this topic; included are all of the articles and many of the illustrations from that issue. A fascinating exposition of hardware, software, time-sharing, applications, artificial intelligence, etc., by such outstanding contributors as McCarthy, Greenberger, Minsky, Oettinger and Suppes.

Digital Computer Programming: Logic and Language. By C. M. Thatcher and A. J. Capato. Addison-Wesley, Reading, Mass., 1967. xi+159 pp. $3.95 (paper).

"Designed for a freshman-level programming course. . . . The approach is based on the conviction that programming *logic* is more important than programming language, and that both logic and language should be learned from a problem-oriented (rather than a computer-oriented) point of view." Internal computer operation is taught through the medium of a hypothetical computer. Mathematical model formulation, programming logic, language characteristics, number storage schemes, assembly and compiler software, Fortran I, II, IV and other languages are covered. A leisurely treatment.

Limits and Continuity. By T. C. J. Leavitt. McGraw-Hill, New York, 1967. vii+177 pp. $3.50 (cloth), $2.50 (paper).

This exceptionally attractive self-study supplement leads the reader from intuitive ideas to precise formulations via careful motivation, conversation, and programmed material. Many students and teachers of calculus (as well as potential authors of elementary analysis texts) should look into this lucid exposition.

Analytic Geometry: A Programmed Text. By T. A. Davis. McGraw-Hill, New York, 1967. ix+438 pp. $6.50 (cloth), $4.50 (paper).

The first unit contains material "usually found in the first 50 or 60 pages of college textbooks on analytic geometry and calculus. . . . The second unit deals with conic sections." Intended as a text for a separate course in analytic geometry or as a supplementary text for a beginning calculus course.

Lectures on Calculus. Edited by K. O. May. Holden-Day, San Francisco, 1967. vii+180 pp. $6.50 (paper).

Unfortunately, calculus "remains one of the least exciting courses taken by the undergraduate student, perhaps because the great ideas are obscured by routine manipulations and obsolete applications that have been incorporated in the courses over the years. Yet even the elementary course in calculus opens the doors to many treasure-houses of interesting ideas." New approaches and novel applications are prominent in this collection of lectures by Copeland, Mancil, Richmond, Sagan, Guggenheimer, Wilansky, Munroe, Wyler, and Fort.

Vectors. By J. A. Hummel. Addison-Wesley, Reading, Mass., 1965. vii+108 pp. $1.95 (paper).

"An attempt is made to give a rigorous development of the elementary theory of three-dimensional vectors. . . . Although vectors are defined in terms of their coordinates (in an intrinsic coordinate system), there is actually a coordinate-free emphasis in the development. . . . Each newly introduced concept is investigated in terms of its similarity to and its differences from the properties of the real number system. . . . " Considerable use is made of geometric intuition; the only prerequisites are intermediate algebra and trigonometry.

About Vectors. By Banesh Hoffman. Prentice-Hall, Englewood Cliffs, New Jersey, 1966. ix+134 pp. $3.25 (paper).

"This book is written as much to disturb and annoy as to instruct. Indeed, it seeks to instruct primarily by being disturbing and annoying, and it is often deliberately provocative. . . . It is intended as a supplement and corrective to textbooks, and as collateral reading in all courses that deal with vectors." A great deal of originality is reflected in this illuminating exposition; contains numerous exercises and a commendable blend of theory and applications.

Percentage Baseball, 2nd ed. By Earnshaw Cook (in collaboration with W. R. Garner). MIT Press, Cambridge, Mass., 1966. xiii+417 pp. $3.95 (paper).

The first (hardback) edition was reviewed in this MAGAZINE, 39 (1966) 303–304.

Remarks on the Foundations of Mathematics. By Ludwig Wittgenstein. MIT Press, Cambridge, Mass., 1967. xix+203 pp. $3.45 (paper).

First published in 1956; this edition actually contains over 400 pages: each page in German is opposite its translation into English.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before December 1, 1968.

PROPOSALS

698. *Proposed by Bertrand N. Bauer, Northwestern University.*

What is the probability of "qualifying" in the dice game "Ship-Captain-Crew"? The game is played with five dice. Each player, in his turn, throws all dice, or any portion of them he wishes, a maximum of three times. To qualify, a player must throw a 6 on one die (the "Ship"), a 5 on a second die (the "Captain") and a 4 on a third die (the "Crew"). He must throw them in that order, although getting them simultaneously is permitted.

699. *Proposed by James G. Seiler, San Diego City College.*

Find an oblique Heronic triangle with sides of three nonzero digits each, such that the nine digits involved are distinct. A Heronic triangle is defined as one that has integers for the lengths of the sides and also an integer representing the area.

700. *Proposed by Charles W. Trigg, San Diego, California.*

Find a prime number which is the sum of primes in two ways such that either set of addends and the sum together contain no duplicated digits.

701. *Proposed by Gregory Wulczyn, Bucknell University.*

Martin Gardner defines a w-digit automorph, m, to be an integer whose square has the w digits of m at its tail end.

1. Prove that if m is a w-digit automorph with base r, then $(1\,0 \cdots 0\,1)_r - m_r$ is also a w-digit automorph where $1\,0 \cdots 0\,1$ contains $w-1$ zeroes.

2. If $t>1$, $s_i \geqq 1$ how many automorphs does the base number $r_1^{s_1} = p_2^{s_2} \cdots p_t^{s_t}$ have?

702. *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

If the line joining the circumcenter and orthocenter of a triangle ABC is parallel to a side, prove that $\tan A$, $\tan B$, and $\tan C$ are in arithmetic progression in some order.

703. *Proposed by A. H. Beiler, Brooklyn, New York, and J. A. H. Hunter, Toronto, Canada.*

Prove that for integral $n \geqq 1$, $(x+1)^n - x^n - 1$ is divisible by x^2+x+1, if and only if $n = 6m \pm 1$.

704. *Proposed by Maxey Brooke, Sweeny, Texas.*

The adjoint matrix of

$$\begin{bmatrix} I & A & -A \\ A & R & A \\ T & A & T \end{bmatrix}$$

is

$$\begin{bmatrix} X & -R & I \\ M & AM & -X \\ -X & -T & AA \end{bmatrix}$$

What digits are represented by each letter?

SOLUTIONS

Late Solutions

H. S. Hahn, West Georgia College: **662**; *Richard A. Jacobson, Houghton College, New York:* **656**; *K. L. Singh, Memorial University of Newfoundland:* **632**; *Thomas Shewczyk, University of Wisconsin at Waukesha:* **669** (two solutions); *Susan G. Takach, University of Maryland:* **669**; *and, Dimitrios Vathis*; *Agios Nicoloios, Chalcis, Greece*; **667**.

Erratum

The last line of the solution to Problem 664, Page 97, March, 1968, should read: $\cdots = (p_n/q_n) + (s_n/q_n) \underset{n\to\infty}{\longrightarrow} x + 0 = x$.

The Tax Rate

677. [January, 1968] *Proposed by Charles R. Wall, University of Tennessee.*

Various states have sales tax rates of 0, 2, 2.25, 2.50, 3, 3.50, 4, and 5 percent. Knowing these rates, what is the smallest amount (before tax) one would have to spend to determine the tax rate:

(1) in a single purchase?

(2) in as many purchases as desired?

Solution by Zalman Usiskin, University of Michigan.

We assume that sales tax is computed to the nearest cent, and that half-cents are always rounded to the higher amount.

(1) Any purchase over $2 differentiates the 2.50 tax rate from the higher rates, for each $\frac{1}{2}$ percent increase in tax rate adds at least a penny to the tax. The problem is to differentiate between 2, 2.25, and 2.50. The following table indicates the amounts where a change occurs in the tax for each of the tax rates, from a tax of .01 to a tax of .10.

	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
2	25	75	125	175	225	275	325	375	425	475
2.25	23	67	112	156	200	245	289	334	378	423
2.50	20	60	100	140	180	220	260	300	340	380

From the table it is seen that \$2.20 is the minimum amount which differentiates between the three tax rates. Since the 5 percent sales tax on \$2.20 amounts to \$0.11, one would have to be prepared to spend \$2.31 to differentiate between the rates.

(2) The smallest amount with any number of purchases is \$.66, including sales tax. The procedure which should be used follows:

1. Spend \$0.17. If tax is 0, then rate is 0, 2, 2.25, or 2.50. Go to 2A.
If tax is \$0.01, then rate is 3, 3.50, 4, or 5. Go to 2B.

2A. Spend \$0.23. If tax is 0, then tax rate is 0 or 2. Go to 3A.
If tax is \$0.01, then tax rate is 2.25 or 2.50. Go to 3B.

2B. Spend \$0.13. If tax is 0, then rate is 3 or 3.50. Go to 3C.
If tax is \$0.01, then rate is 4 or 5. Go to 3D.

3A. Spend \$0.25. If tax is 0, then tax rate is 0.
Total cost: \$0.65
If tax is \$0.01, then tax rate is 2.
Total cost: \$0.66

3B. Spend \$0.20. If tax is 0, then rate is 2.250.
Total cost: \$0.61
If tax is \$0.01, then rate is 2.50.
Total cost: \$0.62

3C. Spend \$0.15. If tax is 0, then tax rate is 3.
Total cost: \$0.46
If tax is \$0.01, then tax rate is 3.50.
Total cost: \$0.47

3D. Spend \$0.10. If tax is 0, then rate is 4.
Total cost: \$0.42
If tax is \$0.01, then rate is 5.
Total cost: \$0.43

Consequently, the tax rate can be determined with purchases and potential tax of \$0.66.

Also solved by George F. Corliss, College of Wooster, Ohio (partially); John Hudson Tiner, Harrisburg, Arkansas; and the proposer. Two incorrect solutions were received.

Tromino Fault Lines

678. [January, 1968] *Proposed by Charles W. Trigg, San Diego, California.*

From an 8 by 8 checkerboard, the four central squares are removed.

a) Show how to cover the remainder of the board with right trominoes so as to have no fault line, or exactly two fault lines, or three fault lines.

b) Show that no covering with right trominoes can have four fault lines.

A right tromino is a nonrectangular assemblage of three adjoining squares. A fault line has its extremities on the perimeter so that a portion of the configuration may be slid along it in either direction without otherwise disturbing the relative position of its parts.

Solution by Benjamin L. Schwartz, The MITRE Corporation, Virginia.

The problem as stated leaves unsettled the question of the existence of exactly one fault line. In the discussion below, we shall also resolve this question affirmatively with an example.

Notation. Denote the 8 horizontal rows of squares by letters A through H. Denote the vertical files by numbers 1 through 8. Denote a line between rows by the names of the two bordering rows (e.g., line BC).

(a) The diagrams (1), (2), (3) and (4) show examples with zero, one, two and three fault lines, respectively. In (2), the fault line is 34. In (3), the fault lines are DE and 45; and, in (4), they are CD, EF and 45. As shown later, these arrangements of fault lines are essentially unique.

(b) To prove other cases impossible, we introduce a few easily proved lemmas about coverings with trominoes. Proofs are omitted.

I A $2\times n$ area can be covered iff $3|n$.

II Two adjacent lines (e.g., CD and DE) cannot both be fault lines.

III A 3×3 area cannot be covered exactly.

IV A fault line cannot occur adjacent to an edge.

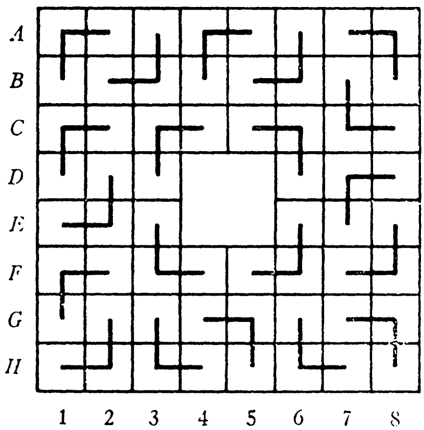
Lemma IV eliminates AF, GH, 12 and 78 as candidates. But Lemma I also eliminates BC, FG, 23 and 67. Hence the only horizontal candidates are CD, DE and EF; and verticals 34, 45 and 56. Furthermore, by Lemma II, if DE is a fault line, it is the only horizontal one. Similarly, if 45 is a vertical fault line, it is the only one.

Thus, the only prospect for a 4-fault-line configuration requires that these lines be CD, EF, 34 and 56. But this means the 3×3 areas in the four corners must be covered exactly, which violates Lemma III.

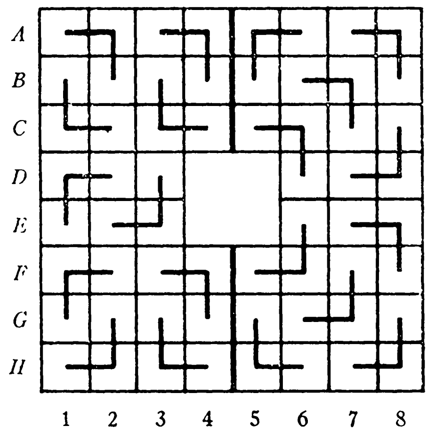
An additional question is whether the arrangement of fault lines in each of the various cases is essentially unique. The affirmative answer to this follows from the following theorem:

If CD is a fault line, so is EF. (Similarly 34 and 56.) To see this, suppose there is a covering with CD as a fault line. Then consider how square $D1$ could be covered. Place a tromino to cover it with each of the three possible orientations of the other two squares. It follows that either the other squares of files 1 and 2 in rows D and E cannot be covered, or can only be covered in such a way as to fill exactly the 2×3 rectangle $(123)\times(DE)$. The same argument applies to $(678)\times(DE)$. Hence EF must be a fault line.

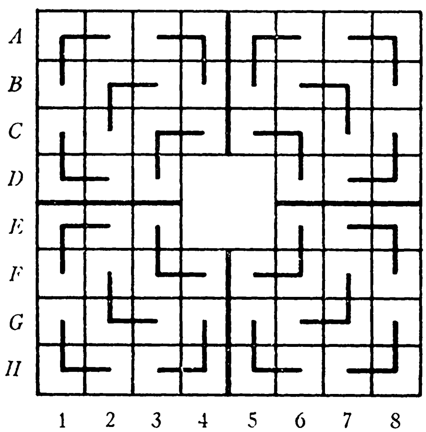
A more laborious and detailed analysis concerns the actual arrangement of the trominoes in the coverings. It has shown that except for "flipover" of the coverings of 2×3 rectangles, and rotations of the entire board, the solutions of (2), (3) and (4) are unique. A similar statement is believed to hold for (1), but has not yet been proved.



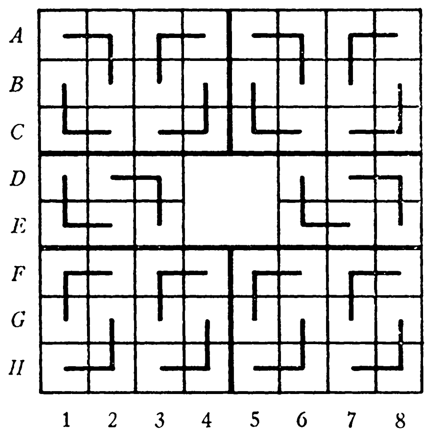
(1)



(2)



(3)



(4)

Also solved by Richard A. Jacobson, Houghton College, New York; Michael J. Smithson, Bellevue, Washington; Zalman Usiskin, University of Michigan; and the proposer.

An Inequality

679. [January, 1968] *Proposed by Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania.*

Prove that $2^\alpha < 1+\alpha$ for $0<\alpha<1$.

I *Solution by Dale Woods, Northeast Missouri State College.*

It is quite well known that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(p-m+1)}{m!}x^m$$

is absolutely convergent if $n>0$ but is not an integer for the interval $[-1, 1]$.

Therefore if $x=1$, $n=\alpha$ $2^\alpha<1+\alpha$, $0<\alpha<1$ since $(\alpha(\alpha-1)/2!)$ is negative, the series is alternating and $|a_n|>|a_{n+1}|$.

II *Solution by David N. Mpongo, Albany State College, Georgia.*

We shall prove this by contradiction.
Suppose

$$2^x \geqq 1+x \qquad \text{for all } 0<x<1.$$

Then

$$x \ln 2 \geqq \ln(1+x),$$

or

$$\ln 2 \geqq \frac{1}{x}\ln(1+x)$$
$$= \ln(1+x)^{1/x}.$$

Taking the limit as x tends to zero, we get

$$\ln 2 \geqq \lim \ln(1+x)^{1/x}$$
$$= \ln \lim(1+x)^{1/x}$$
$$= \ln e$$
$$= 1.$$

But, that $\ln 2 \geqq 1$ is certainly a contradiction.
Hence, it must be true that

$$2^x < 1+x \qquad \text{for all } 0<x<1.$$

III *Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennslyvania.*

$2^0=1+0$ and $2^1=1+1$. In between, we have 2^α convex downwards and therefore below $1+\alpha$.

IV *Solution by Wray G. Brady, University of Bridgeport, Connecticut.*

Let $f(x)=(1+x)^\alpha$ by the extended law of the mean we have:

$$(1+x)^\alpha = 1+\alpha x+R_n(x,\xi)$$

when $R_n(x,\xi)=(\alpha(\alpha-1)/2)x^2(1+\xi)^{\alpha-2}$ and $0<\xi<x$. Since $\alpha-1<0$ it follows $R_n(1,\xi)<0$ and the inequality follows.

V *Solution by J. R. Hanna, University of Wyoming.*

The series in

$$(1) \qquad (1+\alpha)2^{-\alpha} = (1+\alpha)\sum_{j=0}^{\infty}\frac{(-\alpha)^j \log^j 2}{j!}$$

is absolutely convergent for all real α.

Let us write (1) in the form

$$(2)\qquad \begin{aligned}(1+\alpha)2^{-\alpha} &= 1-\alpha\log 2+\sum_{j=2}^{\infty}\frac{(-\alpha)^j\log 2}{(j-1)!}+\alpha+\sum_{j=2}^{\infty}\frac{(-1)^{j-1}\alpha^j\log 2}{(j-2)!}\\ &= 1+\alpha(1-\log 2)\ +\log 2\sum_{j=2}^{\infty}\frac{(-1)^j\alpha^{j+1}}{j(j-2)!}\,.\end{aligned}$$

In (2), the right member converges to a value greater than 1 for $0<\alpha<1$. Therefore $1<(1+\alpha)2^{-\alpha}$ which implies that $2^\alpha<1+\alpha$.

This problem is a special case of the inequality established in *Analytic Inequalities* by Kazarinoff, pp. 21–22. The inequality is stated as part of a theorem:

If $x\geqq -1$ *and* $0<\alpha<1$, *then* $(1+x)^\alpha\leqq 1+\alpha x$.

Equality holds if $x=0$. Thus if $x=1$, $2^\alpha<1+\alpha$. Kazarinoff proves the theorem for α rational only.

Another inequality

$$x^r-1<r(x-1)\qquad (0<r<1,\ x>0,\ x\neq 1)$$

appears in *Inequalities* by Hardy, Littlewood and Pólya, p. 40–42. If $x=2$, the latter becomes $2^r-1<r$ or $2^r<1+r$. The last reference includes a procedure which leads to a proof if r is either rational or irrational.

Also solved by Miguel Bamberger, Albuquerque, New Mexico; Leon Bankoff, Los Angeles, California; Merrill Barnebey and M. S. Arora, (Jointly), Wisconsin State University at LaCrosse; Gladwin E. Bartel, Washington State University; Donald Batman, MIT Lincoln Laboratory; Richard J. Bonneau, Holy Cross College, Massachusetts; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Nicholas C. Brystrom, Northland College, Wisconsin; Robert W. Chilcote, Bedford High School, Ohio; Noel A. Childress, University of Mississippi; George F. Corliss, College of Wooster, Ohio; Santo M. Diano, Philadelphia, Pennsylvania; Glenn L. Dobson, Jr., University of Tennessee at Martin; David Fettner, The City College of New York; Thomas Flynn, St. Mary's College, California; Melvin Friske, University of Wisconsin at Waukesha; Anton Glaser, Pennsylvania State University at Abington; Marc Glucksman, University of California at Los Angeles; John O. Herzog, Pacific Lutheran University, Washington; James C. Hickman, University of Iowa; James M. Howard, Ferris State College, Michigan; William D. Jackson, SUNY at Buffalo; L. A. Jacobson, MIT Lincoln Laboratory; Richard A. Jacobson, Houghton College, New York; Erwin Just, Bronx Community College; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; Joseph D. E. Konhauser, Macalester College, Minnesota; Lew Kowarski, Morgan State College, Maryland; J. F. Leetch, Bowling Green State University, Ohio; Peter A. Lindstrom, Union College, New York; R. S. Luthar, University of Wisconsin at Waukesha; Charles McCracken, Martin-Marietta Corp., Florida; Ben W. Miller, Austin College, Texas; M. Morucci, Wheeling College, West Virginia; Maurice Nadler, Pace College, New York; Melvin Nyman, Ferris State College, Michigan; Frank J. Papp, University of Delaware; Bryan Powers, Del Mar College, Texas; Bob Prielipp, Wisconsin State University at Oshkosh; Stephen K. Prothero, Willamette University, Oregon; Charles L. Ragan, Jr., Bowling Green State University, Ohio; Kenneth A. Ribet, Brown University; Henry J. Ricardo, Yeshiva University, New York; Lawrence A. Ringenberg, Eastern Illinois University; Bernard Rosman, Richard Michaud, Sheldon Kovitz and Natalie Richstone, Boston, Massachusetts (Jointly); W. Alan Sentance, The City University of London, England; Thomas Shewczyk, University of Wisconsin at Waukesha; John Schroetor, South Dakota State University; E. P. Starke, Plainfield, New Jersey; Gary E. Stevens, Bowling Green State University, Ohio; Dimitrios Vathis, Chalcis, Greece; J. S. Vigder, Defence Research Board of Canada; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Bucknell University, Pennsylvania; and the proposer.

A Circular Locus

680. [January, 1968] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C, tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

Solution by the proposer.

Let the ellipse be given by the equation

$$x^2/a^2 + y^2/b^2 = 1 \tag{1}$$

Denoting the center and radius of (C) by (α, β) and r, from $r = (0, t')$, we have

$$r^2 = (a^2\beta^2 + b^2\alpha^2)/(\alpha^2 + \beta^2); \tag{2}$$

r is also given by

$$(x - \alpha)^2 + (y - \beta)^2 = r^2 \tag{3}$$

such that the normal at $T(x, y)$ of (E) passes through the center C.

CT is an extremal distance of $C(\alpha, \beta)$ to (E). To determine it we use the method of Lagrange multipliers. Let

$$F(x, y) = (x - \alpha)^2 + (y - \beta)^2 + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

where λ is determined by

$$\begin{aligned} 1/2F_x &= x - \alpha + \lambda x/a^2 = 0, \\ 1/2F_y &= y - \beta + \lambda y/b^2 = 0 \end{aligned} \tag{4}$$

and (1). Eliminating x, y, α, β we obtain a quartic equation in λ. So we proceed in a different way. Supposing that the statement is true, we have

$$OC^2 = \alpha^2 + \beta^2 = (a + b)^2 \tag{5}$$

Solving α, β from (4) and setting in (5) and comparing the result with (1) we get $\lambda = ab$.

We observe that if we choose $\lambda = ab$, the three equations (1), (2), (3) are consistent and this consistency gives $\alpha^2 + \beta^2 = (a+b)^2$ proving that the locus of $C(\alpha, \beta)$ is a circle.

Also solved by Michael James Smithson, Bellevue, Washington.

Orthic Similarity

681. [January, 1968] *Proposed by Leon Bankoff, Los Angeles, California.* Find an obtuse triangle that is similar to its orthic triangle.

Solution by the proposer.

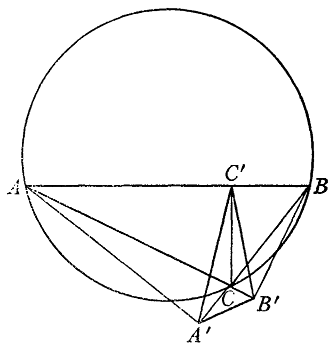
In the obtuse triangle ABC, in which $C > B > A$, let A', B', C' denote the projections of A, B, C upon BC, AC, AB, respectively. In the orthic triangle

$A'B'C'$, we have $B'>A'>C'$, an order of magnitude verified by comparison with the tangential triangle, the sides of which are known to be parallel to those of the orthic triangle. Thus we require $A=C'$, $B=A'$, $C=B'$.

Referring to the cyclic quadrilaterals $AA'CC'$ and $C'CB'B$, and noting that the altitudes of triangle ABC bisect the interior angles of the orthic triangle, we find $B=A'=2A$ and $C=B'=2B$, whereupon $C=2B=4A$. It follows that $A=\pi/7$, $B=2\pi/7$, $C=4\pi/7$.

If the basic obtuse triangle is isosceles, no solution is possible because, with $C>B=A$, we obtain the contradiction $A=B=A'=B'=2A=2B$. Thus the foregoing solution is unique.

It is easy to show that the equilateral triangle is the only acute triangle similar to its orthic triangle.



Also solved by H. S. Hahn, West Georgia College; and Joseph D. E. Konhauser, Macalester College, Minnesota. Two incorrect solutions were received.

Michael Goldberg, Washington, D. C., commented that a degenerate obtuse triangle which had collapsed to a straight line has a similar orthic triangle.

Euler's Limit

682. [January, 1968] *Proposed by John Beidler, Scranton University, Pennsylvania.*

Let b be a fixed integer, $b>1$, and k be a positive integer. Let n be an integer such that the expansion of $n!$ in the base b has kn digits. Find $\lim_{k\to\infty} nb^{-k}$.

Solution by Henry J. Ricardo, Yeshiva University.

Since the expansion of $n!$ in the base b has kn digits, the characteristic of $\log_b n!$ must be $kn-1$. That is,

$$\log_b n! = kn - 1 + \alpha, \qquad 0 \leqq \alpha \leqq 1.$$

Then

$$n! = b^{kn}b^{\alpha-1}, \quad \text{or} \quad b^k = \sqrt[n]{n!}\, b^{(1-\alpha)/n}.$$

Noting that $n \to \infty$ as $k \to \infty$ and using Stirling's asymptotic formula for $n!$, we see that

$$\lim_{k\to\infty} nb^{-k} = \lim_{n\to\infty} n/(\sqrt[n]{n!}\, b^{(1-\alpha)/n}) = e.$$

Also solved by H. S. Hahn, West Georgia College; Kenneth A. Ribet, Brown University; and the proposer.

Equivalent Triangles

683. [January, 1968] *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Two triangles have sides $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$, $\sqrt{c^2+a^2}$ and $\sqrt{p^2+q^2}$, $\sqrt{q^2+r^2}$, $\sqrt{r^2+p^2}$, respectively. Which triangle has the greater area if in addition we have $a^2b^2+b^2c^2+c^2a^2=p^2q^2+q^2r^2+r^2p^2$ and $a>p$, $b>q$?

I *Solution by Michael Goldberg, Washington, D. C.*

The tri-rectangular tetrahedron whose right-angled edges have lengths a, b, c, has the lengths $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$ and $\sqrt{c^2+a^2}$ for its other edges. Since the squares of the areas of the right-triangle faces add to the square of the area of the fourth face, the square of the area of the fourth face is $(a^2b^2+b^2c^2+c^2a^2)/4$. Similarly, for a tri-rectangular tetrahedron whose right-angled edges have lengths p, q, r, the square of the area of the fourth face is $(p^2q^2+q^2r^2+r^2p^2)/4$. But, since we are told that $a^2b^2+b^2c^2+c^2a^2=p^2q^2+q^2r^2+r^2p^2$, the two triangles have the same area, regardless of the relations between a, p, b and q.

II *Comments and solution by J. S. Vigder, Defence Research Board of Canada.*

Obviously a symmetric relationship between the sides of two triangles either tells nothing about which area is greater or else establishes that the two areas are equal. Also the relationship $a>p$, $b>q$ can give no information since it is possible to start with two triangles with corresponding sides equal and by making incremental changes in the sides give the one having $a>p$ and $b>q$ an area either greater than or less than the other, and at the same time retain any symmetric relationship providing this relationship does not in itself establish that the two areas are equal. In mathematical language, $a>p$, $b>q$ is an "unnecessary and insufficient condition" both for the areas to be equal and for them to be unequal. Thus one can conclude the areas of the two triangles in the problem are equal.

Since reasoning like the above is only used by mathematicians in discussions in theology, politics, sociology, and other nonmathematical subjects, it will not be accepted by a magazine published by and for mathematicians, hence the following alternative solution is offered.

The area of a triangle with sides x, y, z is given by

$$1/4\sqrt{(x+y+z)(x+y-z)(x-y+z)(-x+y+z)}$$
$$= 1/4\sqrt{2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4}.$$

Making the appropriate substitutions we find that the areas of the two triangles are

$$1/2\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

and

$$1/2\sqrt{p^2q^2+q^2r^2+r^2p^q}$$

and thus the areas are equal.

Also solved by Miguel Bamberger, University of New Mexico; Wray G. Brady, University of Bridgeport, Connecticut; Mannis Charosh, New Utrecht High School, New York; Robert W. Chilcote, Bedford High School, Ohio; Santo M. Diano, Philadelphia, Pennsylvania; H. S. Hahn, West Georgia College; James M. Howard, Ferris State College, Michigan; Lawrence A. Ringenberg, Eastern Illinois University; E. P. Starke, Plainfield, New Jersey; Kenneth M. Wilke, Topeka, Kansas; Dale Woods, Northeast Missouri State College; Gregory Wulczyn, Bucknell University, Pennsylvania; and the proposer.

Comment on Problem 657

657. [May, 1967, and January, 1968] *Proposed by C. Stanley Ogilvy, Hamilton College, New York.*

Ship A is anchored 9 miles out from a point O on a straight shoreline. Ship B is anchored 3 miles out opposite a point 6 miles from O. A boat is to proceed from A to some point on the shore, pick up a passenger, and take him to ship B. It costs the boat owner \$1 per mile to run his boat, whether there is a passenger aboard or not. Where should the owner contract to pick up the passenger so that his net profit (from A to shore to B) shall be a maximum? We can assume that the passenger insists on a straight line course from the pickup point to B.

Comment by the proposer.

The following sentence was inadvertently omitted from the statement of Problem 657: "The boat owner charges \$2 per mile for every mile he carries the passenger." The comment by the proposer [Page 44, January, 1968] does not apply unless this sentence is restored.

The omission rather improves the problem than otherwise. The ten solvers evidently assumed that since no fare was mentioned, the passenger rode free. Under that assumption the word *profit* does not make sense. The statement of the problem implies that a fare is paid. Let f be the amount of the fare in dollars per mile. Take O as the origin and the shore as the x-axis, positive in the direction such that AB intersects the axis where $x=9$. Call this point E. Orient B so that E is to the right of O.

If $f=2$, the boat owner breaks even if he picks up the passenger at $x=-3$. From -3 to E he loses money, to a maximum loss at E; to the left of -3 he gains, to a maximum profit of \$6, approached monotonically as $x\rightarrow-\infty$.

If $2<f<4$, the break-even point approaches E as $f\rightarrow 4$; if $f=4$ he breaks even at E and makes a profit everywhere else; and if $f>4$ he makes a profit everywhere (i.e., no matter where he lands).

For all $f > 2$ the profit maximum grows without bound as $x \to -\infty$.

There is at least $f \simeq 1.82$ below which he cannot make a profit. If $f = 1.82$ he must land at $x = -12.5$ to break even; anywhere else he will lose money.

For $1.82 < f < 2$, the profit function leads to a quartic; there are at least two break-even points (zeroes of the function), with a profit in between but perhaps not everywhere in between.

Comment on Quickie 424

Q424. [January, 1967] Prove that two legs of a right triangle cannot have their lengths equal to twin primes.

[*Submitted by John H. Tiner*]

Comment by Frank Kocher, Pennsylvania State University.

The problem as stated fails to say that the right triangle must have an integral hypotenuse. In addition, it is very easy to show more than the problem asks and more quickly than the given solution. If one realizes that no perfect square is congruent to 2 modulo 4, then it follows that the two legs of a Pythagorean right triangle cannot both be odd. This is a stronger statement than the fact that they cannot be twin primes.

A. Joseph Berlau, Hartsdale, New York, made a similar comment and pointed out that a factor 2 was omitted from the right hand member of the equation $p^2+2p+2=n^2$ in the solution on Page 42.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q435. Evaluate the ratio

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n^2}}{(1-x)(1-x^3)\cdots(1-x^{2n+1})}$$

divided by

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n^2+n}}{(1-x^2)(1-x^4)\cdots(1-x^{2n+2})}$$

[*Proposed by Irving Gerst and M. S. Klamkin*]

Q436. Give an algorithm for finding, for any n, all pairs p, q such that $1/n = 1/p+1/q$.

[*Submitted by Mark Mandelker*]

Q437. If the number of sides of a convex polygon circumscribed about a circle is odd and the length of each side is a rational number, show that the length of each segment into which the sides of the polygon are divided by the points of tangency is a rational number.

[*Submitted by Ned Harrell*]

Q438. Show that $(3n)!/(6^n n!)$ is a positive integer if n is.

[*Submitted by Charles W. Trigg*]

Q439. A student was assigned an $n \times n$ multiplication table to test whether it can be the table of a group. Closure and commutativity were easily checked. When asked how many cases he would need to verify to prove that the associative law holds throughout, he replied:

For the relation $a(bc) = (ab)c$ we can choose a in n ways, b in $n-1$ ways and c in $n-2$ ways. Hence there will be $n(n-1)(n-2)$ cases to test.

His result happens to be correct but the argument is invalid. Correct it.

[*Submitted by E. P. Starke*]

(Answers on page 191)

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1968 recipients of these awards, selected by a committee consisting of Ivan Niven, Chairman; C. W. Curtis, and Edwin Hewitt, were announced by President Moise at the Business Meeting of the Association on August 27, 1968, at the University of Wisconsin. The recipients of the Ford Awards for articles published in 1967 were the following:

Frederic Cunningham, Jr., Taking Limits Under the Integral Sign, this MAGAZINE, 40 (1967) 179–186.

W. F. Newns, Functional Dependence, MONTHLY, 74 (1967) 911–920.

Daniel Pedoe, On a Theorem in Geometry, MONTHLY, 74 (1967) 627–640.

K. L. Phillips, The Maximal Theorems of Hardy and Littlewood, MONTHLY, 74 (1967) 648–660.

F. V. Waugh and Margaret W. Maxfield, Side-and-diagonal Numbers, this MAGAZINE, 40 (1967) 74–83.

H. J. Zassenhaus, On the Fundamental Theorem of Algebra, MONTHLY, 74 (1967) 485–497.

HENRY L. ALDER, *Secretary*

ANSWERS

A435. The ratio is one, since each sum is one. This follows from the known simple summation

$$1 = \frac{1}{1 - a_1} - \frac{a_1}{(1 - a_1)(1 - a_2)} + \frac{a_1 a_2}{(1 - a_1)(1 - a_2)(1 - a_3)} - \cdots$$

A436. It is easy to show that $1/n = 1/(n+a) + 1/(n+b)$ if and only if $ab = n^2$.

A437. Let $a_1, a_2, \cdots, a_n$ designate successive sides of the polygon and let x designate the portion of a_1 from the first vertex to its point of tangency. Then determine each succeeding segment between a vertex and a point of tangency in terms of x. The last section of a_n will equal the first segment of a_1. Thus $x = a_n - a_{n-1} + a_{n-2} - \cdots + a_1 - x$ or $x = \frac{1}{2}(a_n - a_{n-1} + a_{n-2} - \cdots + a_1)$. Hence x is rational. Since this argument will hold for any segment, it follows that each segment is rational.

A438. $N = 3^n n! = 3^n \cdot 1 \cdot 2 \cdot 3 \cdots n = 3 \cdot 6 \cdot 9 \cdots (3n)$. The numbers are one less and one more than the exhibited odd factors of N are even, so $(3n)!$ has n even factors other than those in N. Hence, $(3n)!/2^n N = (3n)!/(6^n n!)$ is a positive integer if n is.

A439. There is no reason why the letters should be distinct. Hence there are really $n \times n \times n$ instances to consider. But there is nothing to prove if one or more of the three elements is the identity. Also there is nothing to prove if the three elements are alike. Hence the total number of instances for which testing is needed is $(n-1)^3 - (n-1) = n(n-1)(n-2)$.

(Quickies on pages 223–224)

ABSOLUTER GEOMETRY

RICHARD MENZEL, Haile Sellassie I University

1. Introduction. A general plane synthetic geometry which we will call "absoluter geometry" may be postulated for which all absolute geometries and all singly elliptic geometries (and perhaps some other geometries) will serve as models.

The incidence axioms employed are those of projective geometry. The notion of separation of four concurrent lines is used as a primitive idea. The axioms regarding this separation are duals of those employed by Borsuk and Szmielew [1, pages 365–6] in their treatment of projective geometry, except for slight further modifications in Axioms 8 and 9 (see Section 2). Nothing is assumed regarding the presence, absence, or number of parallels to a line L which pass through a point not on L. The continuity axiom is that of Dedekind, a plane separation axiom is employed, and the congruence axioms are generalizations of

Q437. If the number of sides of a convex polygon circumscribed about a circle is odd and the length of each side is a rational number, show that the length of each segment into which the sides of the polygon are divided by the points of tangency is a rational number.

[*Submitted by Ned Harrell*]

Q438. Show that $(3n)!/(6^n n!)$ is a positive integer if n is.

[*Submitted by Charles W. Trigg*]

Q439. A student was assigned an $n \times n$ multiplication table to test whether it can be the table of a group. Closure and commutativity were easily checked. When asked how many cases he would need to verify to prove that the associative law holds throughout, he replied:

For the relation $a(bc)=(ab)c$ we can choose a in n ways, b in $n-1$ ways and c in $n-2$ ways. Hence there will be $n(n-1)(n-2)$ cases to test.

His result happens to be correct but the argument is invalid. Correct it.

[*Submitted by E. P. Starke*]

(Answers on page 191)

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1968 recipients of these awards, selected by a committee consisting of Ivan Niven, Chairman; C. W. Curtis, and Edwin Hewitt, were announced by President Moise at the Business Meeting of the Association on August 27, 1968, at the University of Wisconsin. The recipients of the Ford Awards for articles published in 1967 were the following:

Frederic Cunningham, Jr., Taking Limits Under the Integral Sign, this MAGAZINE, 40 (1967) 179–186.

W. F. Newns, Functional Dependence, MONTHLY, 74 (1967) 911–920.

Daniel Pedoe, On a Theorem in Geometry, MONTHLY, 74 (1967) 627–640.

K. L. Phillips, The Maximal Theorems of Hardy and Littlewood, MONTHLY, 74 (1967) 648–660.

F. V. Waugh and Margaret W. Maxfield, Side-and-diagonal Numbers, this MAGAZINE, 40 (1967) 74–83.

H. J. Zassenhaus, On the Fundamental Theorem of Algebra, MONTHLY, 74 (1967) 485–497.

HENRY L. ALDER, *Secretary*